A FACT ABOUT REGULAR LOCAL RINGS

MATH 538 FALL 2011

1. Prime avoidance and regular local rings are domains

We will prove that regular rings are integral domains. Before continuing however, I need a stronger form of prime avoidance.

**Lemma 1.1** (Prime avoidance #2). Suppose that \( R \) is a ring and that \( P_2, \ldots, P_t \subseteq \) are prime ideals and \( P_1 \) is any other ideal that is not necessarily prime. Suppose that \( I \) is an ideal such that

\[ I \subseteq \left( \bigcup_i P_i \right) \]

Then \( I \) is contained in at least one of the \( P_i \).

*Proof.* The proof is by induction. If \( t = 1 \), there is nothing to prove. We now assume it for \( t - 1 \) and so we may assume that \( I \) is not contained in the union of any proper subcollection of the \( P_i \). In particular, we may pick \( x_i \in I \setminus \left( \bigcup_{j \neq i} P_j \right) \) for each \( i = 1, \ldots, t \). Consider

\[ y = (x_1 \cdots x_{n-1}) + x_n \]

noting that \( P_n \) is prime. This element is clearly in \( I \), so it must be in one of the \( P_i \). There are two cases.

1. \( y \in P_n \): Since \( x_n \in I \) and \( x_n \notin P_j \) for \( j \neq n \), we have that \( x_n \in P_n \). Then \( y - x_n = x_1 \cdots x_{n-1} \in P_n \) as well so at least one of the \( x_1, \ldots, x_{n-1} \in P_n \). But that is a contradiction.
2. \( y \in P_j, j \neq n \): Now, \( x_j \in I \) and so \( x_j \in P_j \) and thus \( x_1 \cdots x_{n-1} \in P_j \) as well. So
\[ x_n = (x_1 \cdots x_{n-1}) - y \in P_j \]

which is also a contradiction.

We have proven our result. \( \square \)

**Remark 1.1.** It is actually possible to prove the same result while assuming both \( P_1 \) and \( P_2 \) are NOT necessarily prime.

**Theorem 1.2.** Suppose that \((R, m)\) is a regular local ring. Then \( R \) is an integral domain.

*Proof.* We proceed by induction on the dimension of \( R \) – the case of dimension zero being obvious and the case of dimension 1 being clear from our previous work on DVRs. We thus assume that \( \dim R \geq 1 \). Indeed, set \( P_1, \ldots, P_t \) to be the set of minimal primes of \( R \). We have \( m \supseteq P_i \) by the minimality of the \( P_i \) and the fact that \( R \) is not zero dimensional. By Nakayama’s lemma, we know that \( m \supseteq m^2 \).

Suppose now that \( m \subseteq (m^2) \cup (\bigcup_i P_i) \), but then the previous lemma provides a contradiction and so we may choose \( x \in m \) and \( x \notin P_i \) for any \( i \) and \( x \notin m^2 \).

We have two claims:

1. \( \dim R/\langle x \rangle = (\dim R) - 1 \). We prove this claim. Choose
\[ P_i = Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_d = m. \]

To be a chain of primes of maximal length (in other words, \( \dim R = d \)). This chain of primes is also maximal length in \( R/P_i \). Indeed, by the above, \( x \) is a regular element on \( R/P_i \) and so \( \dim R/(P_i + \langle x \rangle) = \dim(R/P_i) - 1 = \dim R - 1 \). In particular,
dim $R/\langle x \rangle \geq \dim R - 1$. On the other hand, if $\dim R/\langle x \rangle = \dim R$, then we may choose a chain of primes $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_d \subseteq R$ which remains a proper sequence of primes after passing to $R/\langle x \rangle$. But again, we can assume that $Q_0 = P_i$ for some $i$ (since $Q_0$ must be some minimal prime). This sequence must stay a proper sequence of primes after passing to $(R/\langle x \rangle)/Q_0 = (R/\langle x \rangle)/P_i \cong (R/P_i)/\langle x \rangle$. But this is impossible by the above argument. Thus we have proved (i).

(ii) $R/\langle x \rangle$ is regular. This is easy $\dim R/\langle x \rangle = (\dim R) - 1$ and $m/\langle x \rangle$ has $(\dim R) - 1$ generators. This proves (ii).

Our inductive hypothesis then implies $R/\langle x \rangle$ is an integral domain and in particular that $\langle x \rangle$ is prime. In particular, $P_i \subseteq \langle x \rangle$ for some $i$. Choose $y \in P_i$ and write $y = rx$ for some $r \in R$. Since $x \not\in P_i$ and $rx = y \in P_i$, we see that $r \in P_i$. It follows that $x \cdot P_i = P_i$ and so $m \cdot P_i = P_i$. This contradicts Nakayama’s Lemma and completes the proof. □