HOMEWORK # 1
DUE FRIDAY SEPTEMBER 2ND

MATH 538 FALL 2011

1. Suppose that $\phi : R \to S$ is a ring homomorphism and I is an ideal of R. If $\mathfrak{a}$ is an ideal of S, prove that $\phi^{-1}(\mathfrak{a})$ is an ideal of R. More generally, if $\mathfrak{b}$ is an ideal of R and $\mathfrak{a} = \langle \phi(\mathfrak{b}) \rangle_S$ is the ideal of S generated by $\phi(\mathfrak{b})$, then observe that $\mathfrak{b} \subseteq \phi^{-1}(\mathfrak{a})$.

(a) Give an example to show that it is possible that $\mathfrak{b} \neq \phi^{-1}(\mathfrak{a})$.

(b) Suppose that the map $R \to S$ splits as a map of R-modules (in other words, there is an R-module map $\phi : S \to R$ that sends 1_S to 1_R). Prove that then $\mathfrak{b} = \phi^{-1}(\mathfrak{a})$.

2. Let A be a ring and $\mathfrak{n} = \sqrt{0}$ is its nilradical. Show that the following are equivalent:

(i) A has exactly one prime ideal.

(ii) Every element of A is either a unit or nilpotent.

(iii) $A/\mathfrak{n}$ is a field.

3. Suppose that R is a ring with nilradical $\mathfrak{n} = \sqrt{0}$. Suppose that $u \in R$ is a unit and $x \in \mathfrak{n}$ is nilpotent. Prove that $u + x$ is also a unit in R. Similarly show that if $y \in R$ and $(y + \mathfrak{n})$ is a unit in $R/\mathfrak{n}$, then y is a unit in R.

4. Find an example of each of the following:

(i) Prime ideals $P_1, P_2 \subseteq R$ such that $P_1 + P_2$ is both not prime and also not equal to R.

(ii) A ring homomorphism $\phi : R \to S$ and a maximal ideal $\mathfrak{m} \in S$ such that $\phi^{-1}(\mathfrak{m})$ is not a maximal ideal of R.

(iii) Ideals I and J in a ring such that $IJ \neq I \cap J$.

5. Suppose that P is an ideal in R. Prove that the following are equivalent.

(i) P is prime.

(ii) If we are given ideals $I, J \subseteq R$ such that $IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$.

Conclude that if $P \subseteq R$ is prime and we are given ideals $I, J \subseteq R$ such that $I \cap J \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$.

6. Suppose that R is a commutative ring and $R^n \cong R^m$ (the two free modules are isomorphic). Prove that $n = m$ (this is false for non-commutative rings).