WORKSHEET #5 – MATH 536
SPRING

DUE MONDAY, APRIL 7TH

Suppose that R is a Noetherian ring and $M = \langle x_1, \ldots, x_n \rangle$ is a Noetherian (and so finitely generated) R-module. We have a surjective map

$$\rho : R^{\oplus n} \longrightarrow M$$

sending the canonical basis element e_i to the generators. The kernel of this map $K = \ker \rho$ is also a finitely generated R-module $K = \langle y_1, \ldots, y_m \rangle$ (as it is a submodule of a Noetherian module) and so we have another surjection

$$R^{\oplus m} \xrightarrow{\kappa} K \subseteq R^{\oplus n}.$$

We view this as a map

$$\phi : R^{\oplus m} \longrightarrow R^{\oplus n}.$$

By the first isomorphism theorem the cokernel of ϕ is isomorphic to M (in other words $M \cong \mathrm{coker}\phi = R^{\oplus n}/\phi(R^{\oplus m})$).

Viewing the domain and codomain of ϕ as column vectors we can interpret ϕ as a $n \times m$ matrix $A = [a_{ij}]$ with entries in R.

1. Suppose that the matrix A is zero except along the diagonal $i = j$. Describe M as a direct sum of cyclic modules if $m > n$, if $m < n$ and if $m = n$.

2. Suppose we multiply a column of A by a unit in R to obtain a matrix A'. Prove that corresponding cokernel of $R^{\oplus m} \xrightarrow{A'} R^{\oplus n}$ is still isomorphic to M.

Hint: What does the multiplication do to the generators of various modules?

3. Now consider column interchange (switching columns). What does that do to the modules involved?

4. Finally consider column replacement, adding column i times an element $r \in R$ to column j, $\mathbf{a}'_{\bullet j} = r\mathbf{a}_{\bullet i} + \mathbf{a}_{\bullet j}$ to obtain a matrix A'. Show that the cokernel of the map $\phi_{A'} : R^{\oplus n} \longrightarrow R^{\oplus m}$ corresponding to the matrix A' is still isomorphic to M.

Now we turn to row operations, which are somewhat more complicated.

5. Suppose we do a row operation to the matrix A to obtain A' (either row scaling (by a unit), interchange or replacement). Show that while the image of $\phi_{A'}$ can change, that coker$\phi_{A'}$ is isomorphic to cokerϕ_A.

Hint: Doing one of these row operations should be treated the same as performing an isomorphism of $R^{\oplus n}$.

6. Let M denote the cokernel of the matrix

$$\mathbb{Z}^{\oplus 2} \xrightarrow{\begin{bmatrix} 5 & 2 \\ 0 & 3 \\ -1 & -2 \end{bmatrix}} \mathbb{Z}^{\oplus 3}$$

Write M as isomorphic to a direct sum of cyclic R-modules.

7. Suppose $R = k[x]$ where k is a field and M is the R-module given by generators a, b, c modulo the relations $xa + 2b = 0$, $(x + 1)b + c = 0$, $a + (x^2 + x)c = 0$. Write M as a direct sum of cyclic modules.