WORKSHEET #4 – MATH 536
SPRING

NOT DUE

We will define rings of fractions more generally than in the book. The book does this in some sort of hybrid generality, we may as well do it in full generality as it isn't much harder. Throughout this worksheet, R is a commutative ring with unity.

Definition 0.1. A *multiplicative set* $W \subseteq R$ is a subset of R that contains $1 \in R$ and is closed under multiplication.

The elements of W are the ones we are going to invert.

Definition 0.2. Let $W^{-1}R$ denote the set of *formal fractions* $\{r/w \mid r \in R, w \in W\}$ modulo the following equivalence. $r_1/w_1 = r_2/w_2$ if there exists $v \in W$ such that $vw_2r_1 = vw_1r_2$.

We will show that $W^{-1}R$ is a ring. It is the "smallest" ring you get by inverting all of the elements in W.

1. Show that the equivalence above is an equivalence relation.

2. Show that $W^{-1}R$ is a ring under the addition $r_1/w_1 + r_2/w_2 = (w_2r_1 + w_1r_2)/(w_1w_2)$ and multiplication $(r_1/w_1)(r_2/w_2) = (r_1r_2)/(w_1w_2)$.

3. Show that there is a ring homomorphism $R \longrightarrow W^{-1}R$ sending $r \mapsto r/1$. Describe the kernel and show that the homomorphism is injective if R is a domain and $0 \notin W$. (What happens if $0 \in W$?)

4. Let k be a field and $R = k[x,y]/\langle xy \rangle$. Let $W = \{1, x, x^2, x^3, \ldots\}$. Show that $W^{-1}R$ is isomorphic to $k[x, x^{-1}] = \{1, x, x^2, \ldots\}^{-1}k[x]$.

5. Suppose that $I \subseteq R$ is an ideal and W is a multiplicative set. Let $\overline{W}$ denote the image of W in R/I. Show that $\overline{W}^{-1}(R/I) \cong W^{-1}R/(I \cdot W^{-1}R)$ where $I \cdot W^{-1}R$ is the ideal generated by the image of I in $W^{-1}R$.

6. Show that for any prime ideal $P \subseteq R$, $R \setminus P$ is a multiplicative set. The ring $R_P := W^{-1}R$ is called the localization of R at P. Let $R = \mathbb{C}[x]$ and let P be the polynomials which vanish at the origin. Describe R_P as a subset of $\mathbb{C}(x)$? (Can you guess why it's called the localization in this case?)