# NOTES ON A TRICKY PROBLEM FROM CLASS

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This problem was taken from an old University of Wisconsin exam.

**1.** Suppose $G$ is a group, and $|G| = p^n m < \infty$ for some prime $p$ which does not divide $m$. Suppose $P \in \mathrm{Syl}_p(G)$ and $N \trianglelefteq G$ is such that $|G : N| = |P| = p^n$.

**(a)** Prove that $N = \{x \in G \mid p \not| \, |x|\}$.

**Solution:** Certainly $|N|$ is not divisible by $p$ so that if $x \in N$ then $p \not| \, |x|$ hence $\subseteq$ is obvious. Thus assume that $x \in G$ and that $p$ does not divide $|x|$. But now let $\overline{x} = xN$ the coset of $N$ in $G/N$. Note that if $x^k = 1$ then $\overline{x}^k = 1$ and hence $|\overline{x}| \big| \, |x|$. In particular, $p \not| \, |\overline{x}|$. But $|G/N| = p^n$ and hence $\overline{x} = 1 \cdot N \in G/N$. But then $x \in N$ as desired.

**(b)** If every element $y \in G \setminus N$ has order $p$ to a power, then show that $P = N_G(P)$.

**Solution:** If $P \neq N_G(P)$ then there exists a prime $q \big| \, |N_G(P)|$, $q \neq p$, note $q|m$. Thus fix $Q \in \mathrm{Syl}_q(N_G(P)) \subseteq N_G(P) \subseteq G$. Now, note that no element of $Q$ has $p$-power order and so $Q \subseteq N$. Consider the commutator

$$[Q, P] = \{x^{-1}y^{-1}xy \mid x \in Q, y \in P\}$$

which measures the extent to which elements of $Q$ and $P$ commute.

**Claim 0.1.** *$[Q, P] \subseteq N$.*

*Proof of claim.* Consider $x^{-1}y^{-1}xy \in [Q, P]$. Since $x \in Q \subseteq N$ we can write $x^{-1}y^{-1}xy = x^{-1}y^{-1}yx' = x^{-1}x'$ for some $x' \in N$ and so $[Q, P] \subseteq N$. This proves the claim. $\square$

But we also assert that:

**Claim 0.2.** *$[Q, P] \subseteq P$.*

*Proof of claim.* Again take $x^{-1}y^{-1}xy \in [Q, P]$. Since $Q \subseteq N_G(P)$, $Q$ normalizes $P$ and so we can write $x^{-1}y^{-1}xy = y'x^{-1}xy = y'y$ for some $y' \in P$. This is in $P$ and the claim is proven. $\square$

Taking the two claims together we have shown that $[Q, P] \subseteq N \cap P$ but $N \cap P$ is a subgroup of $N$ and $P$, groups of relatively prime order and so $[Q, P] = \{1\}$. Thus $x^{-1}y^{-1}xy = 1$ for all $x \in Q$ and $y \in P$ such shows that $xy = yx$ and thus the elements of $Q$ and $P$ commute. But if $x \in Q$ and $y \in P$ are non-unit elements, then $pq \big| |xy|$ and so $xy \notin N$. But then $xy$ has order $p^a$ for some integer $a$ since $xy \notin N$. This is impossible and we have found our contradiction.