MATH 536 – MIDTERM EXAM

1. Suppose G is a nonabelian group of order 21. What Sylow subgroups of G are normal? How many elements of order three does G have?

Solution: The number of Sylow 3-subgroups is equal to 1 mod 3 and divides 7. Thus there are either 1 or 7 such subgroups. Likewise the number of 7-subgroups is 1 mod 7 and divides 3 and hence there is a unique 7-subgroup.

If there is a unique subgroup of size 3 then we have accounted for $2 + 6 + 1$ elements, the 1 is for the identity. This leaves us with 21-9 $= 12$ elements not of order $1, 3$, or 7. These must be order 21 and so G is cyclic and hence Abelian. Thus there cannot be a unique group of order 3 and so there are 7 of them.

In conclusion the only Sylow subgroup of G is the one of size 7. Further we have $2 \cdot 7 = 14$ elements of order 3.

2. Show that no group of order 72 is simple.

Solution: We factor $72 = 2^3 \cdot 3^2$. Let G be a group of order 72. The number of 3-subgroups is equal to 1 mod 3 and divides 8. Thus there can be 1 or 4 of them. If there is 1, then it is normal and G is not simple. On the other hand if there are 4 of them. G then acts on the set of Sylow-3-subgroups by conjugation and hence we get a non-trivial group homomorphism $\varphi : G \to S_4$? Since $|S_4| = 4! = 24$ we see that $K = \ker \varphi$ is not trivial or equal to G. Thus G has a normal subgroup K and so G is not simple.

3. Let $R = \mathbb{Z}[x]$ and let $I = \langle -2, x^3 + 3x^2 - x + 7\rangle$. Find all the prime ideals of R which contain I.

Solution: The primes of R which contain I are in bijection with the primes of

$$R/I = \mathbb{Z}[x]/\langle -2, x^3 + 3x^2 - x + 7\rangle \cong (\mathbb{Z}/2\mathbb{Z})[x]/\langle x^3 + 3x^2 - x + 7\rangle.$$

For the isomorphism we implicitly used the first isomorphism theorem.

In $(\mathbb{Z}/2\mathbb{Z})[x]$ we see that $x^3 + 3x^2 - x + 7 = x^3 + x^2 + x + 1 = (x + 1)^3$. Any prime of this later ring contains $0 = (x + 1)^3$ and hence contains $x + 1$. But $\langle x + 1\rangle$ is already prime in R/I since the quotient is $\mathbb{Z}/2\mathbb{Z}$. Hence the only prime of R that contains I is the inverse image of $\langle x + 1\rangle \subseteq R/I$. This inverse image is

$$\langle 2, x + 1\rangle = \langle x + 1, -2, x^3 + 3x^2 - x + 7\rangle.$$

4. Find a prime ideal Q in $R = \mathbb{Z}[x]$ such that R/Q has 9 elements.

Solution: Let $Q = \langle 3, x^2 + 2\rangle$. $R/Q \cong (\mathbb{Z}/3\mathbb{Z}[x])/\langle x^2 + 2\rangle$. The element $x^2 + 2$ is irreducible over $\mathbb{Z}/3\mathbb{Z}$ since it doesn't have any roots.

Hence, it is easy to see that the elements of R/Q are just cosets $ax + b + Q$ for $a, b \in \{0, 1, 2\}$. There are $9 = 3 \cdot 3$ possibilities and so R/Q is an integral domain with 9 elements.