WORKSHEET #7

DUE MONDAY, DECEMBER 4TH, IN GRADESCOPE

You may turn in this assignment in a non-cyclic group of up to 4 people.

In this worksheet, we will *exclusively work with Abelian groups.* When working with Abelian groups, we typically use the symbol $+$ for the operation to emphasize the commutativity.

Definition. Suppose $(G,+)$ is an Abelian group. We say that G is *finitely generated* if there exist $x_1,\dots,x_n \in G$ such that every element of G can be written as a sum of the x_i or $-x_i$, possibly with multiplicity. That is, for every $g \in G$, we can write

$$g = a_1x_1 + \cdots + a_nx_n = \sum_i^n a_ix_i$$

for some $a_i \in \mathbb{Z}$. (If we were writing our group multiplicative, this would mean that $g = x_1^{a_1}\cdots x_n^{a_n}$, but we are using additive notation).

1. Suppose $(G,+)$ is an Abelian group and $N \subseteq G$ is a subgroup.[1] Suppose that N and G/N are finitely generated. Prove that G is also finitely generated.

[1]It is automatically normal since G is Abelian.

We use $(\mathbb{Z}, +)$ to denote the infinite cyclic group (up to isomorphism). Then $\mathbb{Z}^n$ is the finitely generated Abelian group

$$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$$

obtained by direct summing $\mathbb{Z}$ with itself n-times. Essentially this is $\mathbb{Z}$-vectors. The you can take the usual $\mathbf{e}_i$ to be the generating set.

2. Show that an Abelian group G is finitely generated if and only if there is a surjective homomorphism $\pi : \mathbb{Z}^n \twoheadrightarrow G$ for some n.

3. Suppose that $\phi : G \to H$ is a surjective map of Abelian groups. If G is finitely generated, then so is H.

4. Show there is a subgroup $N \subseteq \mathbb{Z}^n$ such that $N \cong \mathbb{Z}^{n-1}$ and $\mathbb{Z}^n/N \cong \mathbb{Z}$.

5. Suppose that $G \subseteq \mathbb{Z}^n$ is a subgroup. Prove that G is finitely generated. (By how many elements?)

Hint: Use induction on n. The base case is a statement about cyclic groups which we did early in the semester. For the general case, use the N you found above and consider $G \cap N \subseteq N$, and the image of G in $\mathbb{Z}^n/N$.

6. Show that for every finitely generated Abelian group G, there is a group homomorphism

$$\phi : \mathbb{Z}^m \to \mathbb{Z}^n$$

such that G is isomorphic to $(\mathbb{Z}^n)/\phi(\mathbb{Z}^m)$.

Hint: Given a surjective $\pi : \mathbb{Z}^n \to G$, take the kernel N. That is also a finitely generated Abelian group...

A map $\phi : \mathbb{Z}^m \to \mathbb{Z}^n$ so that $G \cong (\mathbb{Z}^n)/(\phi(\mathbb{Z}^m)) =: \text{coker}\phi$ is called a *presentation of* G.

7. Suppose that $\alpha : \mathbb{Z}^m \to \mathbb{Z}^m$ is an isomorphism. If $\phi : \mathbb{Z}^m \to \mathbb{Z}^n$ is a presentation of G, show that the composition:

$$\phi \circ \alpha : \mathbb{Z}^m \xrightarrow{\cong} \mathbb{Z}^m \to \mathbb{Z}^n$$

is also a presentation of G.

8. Suppose that $\beta : \mathbb{Z}^n \to \mathbb{Z}^n$ is an isomorphism. Show that the composition:

$$\beta \circ \phi : \mathbb{Z}^m \to \mathbb{Z}^n \xrightarrow{\cong} \mathbb{Z}^n$$

is also a presentation of G.

Any time you are given a presentation of G, $\phi : \mathbb{Z}^m \to \mathbb{Z}^n$, we see that ϕ can be represented by an integer matrix A with n rows and m columns (simply view elements of $\mathbb{Z}^m$ and $\mathbb{Z}^n$ as column vectors). In this case we say that A is a *presentation matrix* for G.

9. Suppose

$$A = \begin{bmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_m \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

is a presentation matrix for G ($a_i \in \mathbb{Z}$). Prove that we can write G as a direct sum of cyclic groups. How many infinite cyclic groups appear in this decomposition (at least if the $a_i \neq 0$)?

10. Suppose that an Abelian group G is presented by an $n \times m$ integer matrix A. Show that doing any of the following operations to A yields another presentation matrix for G.

(1) Switching two rows of A.
(2) Switching two columns of A.
(3) Adding an integer multiple of one row of A to another row of A. (Row replacement)
(4) Adding an integer multiple of one column of A to another column of A. (Column replacement)

11. Show that every Abelian group which can be generated by 3 elements is isomorphic to a direct sum of cyclic groups. (The same proof works for any finitely generated Abelian group, I only want to read this one though).

Hint: Use the previous exercises. Take a smallest nonzero element of A up to absolute value. If it divides every element in its row and column, make the rest of that row and column zero by adding multiples of rows/columns to other rows/columns. If not, use division-with-remainder to construct another presentation matrix with an element with even smaller absolute value.

12. Suppose G has presentation matrix:

$$\begin{bmatrix} 1 & 3 \\ -2 & 2 \\ 1 & 0 \end{bmatrix}.$$

Write G as a direct sum of cyclic groups.

13. Suppose $(G, +)$ is an Abelian group with three generators $a, b, c \in G$ subject to the relations $2a + b = 0$, $a - 3c = 0$. Write G as a direct sum of cyclic groups.