WORKSHEET #6

DUE MONDAY, NOVEMBER 13TH, IN GRADESCOPE

You may turn in this assignment in a group of up to 4 people.

Definition. Suppose a group G is acting on a set S. This action is called *transitive* if for every $x, y \in S$, there exists a $g \in G$ such that $g * x = y$.

1. Prove that a group action G on S is transitive if and only if there is only one orbit.

2. Suppose G acts on S, $x \in S$ and $H \subseteq G$ is the stabilizer of x. Suppose $g_1, g_2 \in G$. Show that $g_1 * x = g_2 * x$ if and only if $g_1 H = g_2 H$.

3. Suppose G acts on S, $x \in S$ and $H \subseteq G$ is the stabilizer of x. Show that gHg^{-1} is the stabilizer of $g * x$.

4. Suppose G acts on S, $x \in S$ and $H \subseteq G$ and $O \subseteq S$ are the stabilizer and orbit of x respectively. Consider the function

$$\epsilon : G/H \to O$$

which sends gH to $g * x$. Prove that this function is well defined and bijective.

Problem **#4.** *is called the "orbit-stabilizer theorem".*

5. Suppose G acts on S, $x \in S$ an G_x and O_x are the stabilizer and orbit of x respectively. Prove that

$$|G| = |G_x| \cdot |O_x|.$$

Hint: Use Lagrange's theorem and the orbit-stabilizer theorem.

6. Let G denote the group of symmetries of a cube in $\mathbb{R}^3$ (that is, symmetries of the cube defined by orthogonal 3×3 matrices). The cube has 6 faces $F_1, F_2, \ldots, F_6$ and so G also acts on the set of faces. (Note, an element of G might send a face back to itself even if it rotates that given face, ie rotations sometimes stabilize a given face).

(a) How many elements are in the stabilizer of F_1?

(b) How many elements are in the orbit of F_1?

(c) Use the previous two parts to count the number of elements of G.

7. Use the same strategy as in **6.** to compute the number of symmetries of the icosohedron (a 20-sided regular polyhedron with triangular faces).

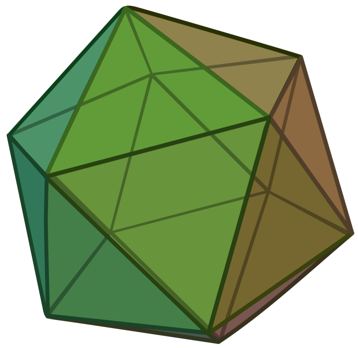

Image taken from wikipedia.