WORKSHEET #4

DUE TUESDAY, OCTOBER 3RD, IN GRADESCOPE

You may turn in this assignment in a group of up to 4 people.

Suppose that you are given two groups A and B. Define a new group, $A \oplus B$ as follows. The elements of $A \oplus B$ is the set of pairs

$$\{(a, b) | a \in A, b \in B\}.$$

The operation is defined as follows $(a, b)(a', b') = (aa', bb')$.

1. Show that $A \oplus B$ is a group. Further prove that $A \oplus B$ is Abelian if and only if A and B are both Abelian.

2. Suppose that A and B are groups of finite order. Show that $|A \oplus B| = |A||B|$. Further show that $A \oplus B$ has a subgroup H with order $|A|$ and a different subgroup K of order $|B|$ such that $H \cap K = \{e_{A \oplus B}\}$.

3. Suppose that A and B are finite cyclic groups $|A| = n$ and $|B| = m$.

(a) If $n = 2$ and $m = 3$, prove that $A \oplus B$ is cyclic.
(b) If $n = 2$ and $m = 2$, prove that $A \oplus B$ is not cyclic.
(c) Is $A \oplus B$ cyclic if $n = 4$ and $m = 6$?
(d) Find a condition on n and m which completely characterizes the n and m such that $A \oplus B$ is cyclic. Prove your condition is correct.

4. Suppose that A and B are infinite cyclic groups. Prove that $A \oplus B$ is not cyclic.

5. Suppose H is a subgroup of a group G and that H is *not* normal. Prove that there are left cosets aH and bH such that

$$(aH)(bH) = \{ahbh' \mid h, h' \in H\}$$

is not a left coset.

6. Let G be the group of matrices of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ where $a, d \neq 0$. For each of the following subsets S,

(i) Determine if they are a subgroup.
(ii) Determine if they are a normal subgroup (if S is a subgroup).
(iii) If S is a normal subgroup, identify the quotient group G/S (try to find an isomorphic group you are familiar with).

(a) S is the subset defined by $b = 0$.

(b) S is the subset defined by $d = 1$.

(c) S is the subset defined by $a = d$.