WORKSHEET #3

DUE WEDNESDAY, SEPTEMBER 20TH, IN GRADESCOPE

You may turn in this assignment in a cyclic group of up to 4 people.

Consider the finite set $\Gamma_n = \{1, \ldots, n\}$ and let S_n denote the group of bijections from Γ_n to itself, where the operation is composition. There is a nice notation for these functions that is quite common.

If we write (1572) that is the function that sends, $1 \mapsto 5$, $5 \mapsto 7$, $7 \mapsto 2$ and $2 \mapsto 1$. All the other numbers are just sent back to themselves.

$$(\quad 1 \quad 5 \quad 7 \quad 2 \quad)$$

Such a function is called a *cycle* (in this case a *4-cycle* since there are 4 numbers). Note we could have also written the cycle (1572) as (5721), but for convenience, we always start the cycle notation with the smallest number that it acts nontrivally on.

We can compose two cycles, to get another function, for example, consider the function:

$$(132)(24) := (132) \circ (24)$$

We first plug in $x = 1$, to (24), which sends it to 1. Then we plug 1 into (132), which sends it to 3. Doing this for $x = 1, 2, 3, 4$ we see that

$$\begin{aligned} 1 \mapsto 1 \mapsto 3 \\ 2 \mapsto 4 \mapsto 4 \\ 3 \mapsto 3 \mapsto 2 \\ 4 \mapsto 2 \mapsto 1 \end{aligned}$$

This can be rewritten as the cycle (1324). In other words

$$(132)(24) = (1324)$$

Definition. Two cycles $(a_1 a_2 \ldots a_s)$ and $(b_1 b_2 \ldots b_t)$ are called *disjoint* if none of the a_i equal any of the b_j. For example

$$(12) \text{ and } (23) \text{ are not disjoint}$$

but

$$(12) \text{ and } (34) \text{ are disjoint.}$$

1. Compute (24)(132) and verify it is not the same as (1324) = (132)(24).

2. If $(a_1a_2\dots a_s)$ and $(b_1b_2\dots b_t)$ are disjoint, prove that their product (ie composition) commutes:

$$(a_1a_2\dots a_s)(b_1b_2\dots b_t) = (b_1b_2\dots b_t)(a_1a_2\dots a_s).$$

3. Prove that every element $\alpha = S_n$ can be written as a product of disjoint cycles.

Hint: Start with an element, and follow it around wherever α^i sends it and make a cycle. Take the next element that hasn't appeared yet, and repeat.

4. Let T be a set. We know that $\mathrm{Perm}(T) = S_T$, the set bijections from T to itself (under composition), is a group. Suppose that T is a finite set with n elements, prove that $\mathrm{Perm}(T)$ is isomorphic to S_n.

5. Suppose G is a finite group of order n. For each element $g \in G$ consider the functions:

$$\phi_g : G \xrightarrow{x \mapsto gx} G \quad \text{and} \quad \psi_g : G \xrightarrow{x \mapsto xg} G$$

Consider the maps $\Phi : G \xrightarrow{g \mapsto \phi_g} \mathrm{Perm}(G)$ and $\Psi : G \xrightarrow{g \mapsto \psi_g} \mathrm{Perm}(G)$. Prove that Φ is a group homomorphism but that Ψ is not necessarily a homomorphism.

6. Prove that Φ is injective. Conclude that every group is isomorphic to a subgroup of $\mathrm{Perm}(G)$ and that every finite group is isomorphic to a subgroup of S_n for some $n > 0$.

7. Prove that $\mathrm{Aut}(G)$, the set of isomorphisms from G to itself, a subgroup of $\mathrm{Perm}(G)$ (in particular, it is a group, so you need closure and inverses).

8. For each g in G, we have an automorphism $\mathrm{conj}_g : G \to G$ defined by $\mathrm{conj}(x) = gxg^{-1}$. Prove that the function $C : G \to \mathrm{Aut}(G)$ defined by $g \mapsto \mathrm{conj}_g$ is a homomorphism.

9. Prove that the kernel of $C : G \to \mathrm{Aut}(G)$ from the previous exercise is the *center of* G:

$$Z(G) = \{g \in G \mid x = gxg^{-1} \text{ for all } x \in G\}.$$