WORKSHEET #1 MATH 5310

DUE, MONDAY AUGUST 28TH (IN GRADESCOPE)

Only *ONE* worksheet should be submitted per group, of up to 4, in gradescope. When you submit it, you can list the other members of your group.

A *law of composition* or a *binary operation* on a set S is, formally, a function

$$S \times S \longrightarrow S$$

Informally, it is just a way to combine two elements in a set (like adding vectors). We can denote this operation using any symbol we like, perhaps by no operation or by ☺.

Definition. A *group* is a set G with a law of composition/binary operation (perhaps denoted by ☺) which is:

(1) associative,
(2) has an identity e, or it might be called 1 or 0, ($e☺a = a☺e = a$ for every $a \in G$),
(3) such that every element of G has an inverse (for each $a \in G$, there exists $b \in G$ such that $a☺b = b☺a = e$).

Below are sets with potential laws of composition / binary operations (ie, a way to combine elements). Determine if each set is (or is not) a group and prove your answer. If it is a group, is the operation commutative? [1]

In each problem, to check whether something is a group, you need to show all of the following.

(a) Is the operation a law of composition?
(b) Is the operation associative?
(c) Is there an identity?
(d) Does every element have an inverse?

To show that something is not a group, you only need to show that one thing fails.

1. The numbers $\{0, 1, 2, \ldots, 11\}$. The binary operation is multiplication modulo[2] 12, denoted $\cdot_{12}$. For example, $5 \cdot_{12} 7 = 11$ since $5 \cdot 7 = 35$ and 35 has remainder 11 when dividing by 12.

[1]If the operation is commutative, the group is called *Abelian*.

[2]Meaning divide by n and take the remainder

2. The set of real-valued 2×2 matrices with positive determinant. The operation is matrix multiplication.

Hint: $\det(AB) = \det(A)\det(B)$.

3. The set of real-valued 3×3 matrices with integer (whole number) determinant. The binary operation is matrix multiplication.

4. Fix a group G and an element $a \in G$. Consider the set $H = \{g \in G \,|\, ga = ag\}$. The binary operation on H is the binary operation from the group G.

5. The numbers $\{1, 2, 3, 4, 5\}$. The (potential) binary operation is mutliplication modulo 6.

6. The set of bijective[3] functions $f : \{A, 7, \smiley\} \longrightarrow \{A, 7, \smiley\}$. The operation is function composition.

7. The set of functions $g : \{A, 7, \smiley\} \longrightarrow \mathbb{R}$. The operation is addition of functions. That is, $g + g'$ is defined to be the function which sends $x \mapsto g(x) + g'(x)$, in other words $(g + g')(x) = g(x) + g'(x)$.

[3]that is, one-to-one and onto