WORKSHEET # 4

MATH 435 SPRING 2012

We first recall some facts and definitions about cosets. For the following facts, G is a group and H is a subgroup.

(i) For all $g \in G$, there exists a (left) coset aH of H such that $g \in H$. (One may take $a = g$).
(ii) Cosets are equal or are disjoint. In other words, if $aH \cap bH \neq \emptyset$, then $aH = bH$.
(iii) Properties (i) and (ii) may be summarized by saying: "The (left) cosets of a subgroup partition the group."
(iv) If H is finite, then $|H| = |aH|$ for every coset aH of H (this holds for infinite cosets too).
(v) Cosets of H are generally NOT subgroups themselves.
(vi) Two cosets aH and bH are equal if and only if $b^{-1}a \in H$.
(vii) The subgroup H is called *normal* if $aH = Ha$ (in other words, if the left and right cosets of H coincide, this does not mean $ah = ha$ for all $h \in H$, but it does mean that for all $h \in H$, there exists another $h' \in H$ such that $ah = h'a$).

1. Consider the group $G = \mathbb{Z}$ under addition with subgroup $H = 4\mathbb{Z}$. Write down the three cosets of H.

2. With the same setup as the first problem, consider the cosets $1 + H$ and $2 + H$. If you add these two cosets together, what do you get (in other words, add all possible combinations of elements from each coset)? Write down a general formula for the sum of $n + H$ and $m + H$.

3. Prove that for any integer n, the cosets of $n\mathbb{Z} \subseteq \mathbb{Z}$ form a cyclic group under addition.

4. Suppose that G is a group and H is a *normal* subgroup (but do not assume that G is Abelian). We will show that the set of cosets of H form a group under the following operation.

$$(aH)(bH) = (ab)H.$$

First however, we need to prove that this is well defined. Suppose that $a'H = aH$ and $b'H = bH$. Prove that

$$(ab)H = (a'b')H.$$

5. Prove that the operation above indeed forms a group.

6. Show that there is a surjective group homomorphism $G \to G/H$ whose kernel is exactly H.

7. Find an example of a group G and a normal subgroup H such that both G and H are non-abelian but G/H is Abelian.