SOME SOLUTIONS TO HOMEWORK #3

MATH 435 – SPRING 2012

Certainly there are many correct ways to do each problem.

#21 on page 75. *Let S be any set having more than two elements. If $s \in S$, we define*

$$H_s = \{f \in A(S) | f(s) = s\}.$$

. Prove that H_s cannot be a normal subgroup of $A(S)$.

Proof. Suppose that r, s, t are distinct elements of S. Suppose first that H_s is normal, so that $aHa^{-1} = H$ for every element $a \in A(S)$. Define $g : S \to S$ as follows.

$$g(x) = \begin{cases} s & \text{if } x = r \\ r & \text{if } x = s \\ x & \text{otherwise} \end{cases}$$

It is clear that $g \in A(S)$. Now, consider $ghg^{-1} \in gH_sg^{-1}$ (note by assumption $h(s) = s$). We observe that

$$(ghg^{-1})(r) = g(h(g^{-1}(r))) = g(h(s)) = g(s) = r.$$

Therefore $H_s = gH_sg^{-1} \subseteq H_r$. On the other hand, consider the function $f : S \to S$ defined by the rule:

$$f(x) = \begin{cases} s & \text{if } x = s \\ u & \text{if } x = r \\ r & \text{if } x = u \\ x & \text{otherwise} \end{cases}$$

Notice that $f(s) = s$ and so $f \in A(S)$. Combining everything we know and have assumed, we see that

$$f \in H_s = gH_sg^{-1} \subseteq H_r.$$

But f is clearly not in H_r since $f(r) = u \neq r$. This is a contradiction. ☐

#1 on page 82. *If G is the group of all non-zero real numbers under multiplication and N is the subgroup of all positive real numbers, write out G/N by exhibiting the cosets of N in G and construct the multiplication table for G/N.*

Proof. There are two cosets, $1N$ and $(-1)N$ made up of the positive real numbers and the negative numbers respectively. The multiplication table is:

	1N	(-1)N
1N	1N	(-1) N
(-1)N	(-1)N	1N

☐

#7 on page 83. *If G is a cyclic group and N is a subgroup, prove that G/N is cyclic.*

Proof. First note that N is normal since G being cyclic implies that G is Abelian (note, the fact that N is Abelian is irrelevant), and so the question makes sense.

Suppose that $\{a^i | i \in \mathbb{Z}\} = \langle a \rangle = G$. Now,

$$G/N = \{bN | b \in G\} = \{a^iN | i \in \mathbb{Z}\} = \{(aN)^i | i \in \mathbb{Z}\} = \langle aN \rangle.$$

This proves that G/N is cyclic generated by aN. □

#11 on page 83. *Suppose that G is a group and $Z = Z(G)$ is the center. Then if G/Z is cyclic, show that G is Abelian.*

Proof. Note first that Z is always a normal subgroup, so the question makes sense. Write $G/Z = \langle aZ \rangle$ since G/Z is cyclic. Now choose $b, c \in G$. We need to prove that $bc = cb$. Because the cosets of Z partition the group G, and those cosets are all of the form $a^i Z$ for some $i \in \mathbb{Z}$, we know that $b \in a^i Z$ and $c \in a^j Z$ for some $i, j \in \mathbb{Z}$. Write $b = a^i z_1$ and $c = a^j z_2$ for some $z_1, z_2 \in Z$. Then

$$bc = (a^i z_1)(a^j z_2) = (a^i a^j)(z_1 z_2) = (a^{i+j})(z_1 z_2) = (a^{j+i})(z_2 z_1) = a^j a^i z_2 z_1 = a^j z_2 a^i z_1 = cb.$$

This repeatedly uses the fact that elements of $Z = Z(G)$ commute with everything. □

#14 on page 83. *If G is an Abelian group of order $p_1 p_2 \dots p_k$ where the p_i are distinct primes, prove that G is cyclic.*

Proof. If $k = 1$ we already know the answer, so we can suppose $k > 1$. We know that G contains an element $a_i \in G$ of order p_i. Consider the element $a = a_1 \dots a_k$. Set $\hat{p}_i = (p_1 \dots p_k)/p_i$. Certainly the order of a divides $p_1 p_2 \dots p_k$ by Lagrange's theorem.

Note that $a^{\hat{p}_i} = e a^{\hat{p}_i} e$. Note that $a^{\hat{p}_i} \neq e$ since p_i does not divide $\hat{p}_i$ (and a_i has order p_i). But this implies that the order of a is bigger than $\hat{p}_i$ for each i. Since the order of a divides the product of all the p_i, this implies that the order of a must equal $p_1 p_2 \dots p_k$. This implies that G is cyclic as desired. □