HOMEWORK #0 (WARM UP) – MATH 435

DUE FRIDAY JANUARY 13TH

(1) What is wrong with the following inductive proof that either all cats are the same color? For example, will will "prove" they are all orange, or all black, or all grey, or etc. We will ignore the possibility of a multi-colored cat.

> "We will show that for any set S of n cats, all cats in S are all the same color. For the base case, we consider a set S_1 with 1 cat in it. Clearly that cat has the same color as itself and so the base case is proven. Now suppose that S is a set of $n+1$ cats (we label these cats C_0 to C_n). Suppose that $A \subset S$ is the set $\{C_0, \ldots, C_{n-1}\}$ and $B \subset S$ is the set $\{C_1, \ldots, C_n\}$. By our inductive hypothesis, all the cats in the set A are all the same color, likewise with the cats in B. Now observe that $C_1, \ldots, C_{n-1}$ are in both sets and so C_0 has the same color as $C_1, \ldots, C_{n-1}$ (since they are all in A) which has the same color as C_n (since they are all in B). Thus all cats in S have the same color. This completes the proof by induction."

(2) Find a formula for $3 + 5 + 7 + \cdots + (2n-1)$ and use inductive reasoning to prove that your formula is correct.

Recall that a function is called *injective* if it is one-to-one and that a function is called *surjective* if it is onto.

(3) Suppose that $f : S \to T$ and $g : T \to U$ are two functions and consider the composition $g \circ f : S \to U$.
  (a) Suppose that $g \circ f$ is surjective, prove that g is also surjective.
  (b) Suppose that g and f are both injective, prove that $g \circ f$ is also injective.
  (c) Give an example of two functions g and f such that g is not injective, f is injective, but $g \circ f$ is injective.

(4) Consider the following proof that there are infinitely many prime natural numbers.

> "Suppose that there were finitely many primes, $p_1, \ldots, p_n$. Consider the new number $m = p_1 \cdot p_2 \ldots p_n + 1$. It is clear that $m > p_i$ for $i = 1, \ldots, n$ and so m is not prime. But $p_i \not| m$ for each $i = 1, \ldots, n$ since $m = p_i(\prod_{j \neq i} p_i) + 1$. Now, every integer m is a product of primes by the fundamental theorem of arithmetic, but no prime divides m, a contradiction.

The proof is correct, but consider the following question inspired by it. If we set $p_1, \ldots, p_n$ to be the first n primes, and define $m = p_1 \cdot p_2 \ldots p_n + 1$, is it true that m is always a prime number?

Either prove that this is correct or provide a counter-example (it is ok to use a calculator).

(5) Prove that the square root of 15 is irrational
Hint: Suppose $\sqrt{15} = a/b$ for some positive $a, b \in \mathbb{Z}$, square both sides and derive a contradiction using unique factorization.

(6) Prove directly that if A and B are 2×2 matrices, then $\det(AB) = \det(A) \cdot \det(B)$.

(7) Find the inverse of the square matrix below or prove it is not invertible:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 8 \end{bmatrix}$$