WORKSHEET # 5

MATH 435 SPRING 2011

Recall that we say that *a group G acts on a set X* if the following two properties are satisfied.

(i) $e.x = x$ for all $x \in X$.

(ii) $a.(b.x) = (ab).x$ for all $a, b \in G$ and all $x \in X$. (Notice that the multiplication ab is multiplication in G, while the "." multiplication is the group action).

Now fix $x \in X$. We recall that the *orbit of* x, denoted $\mathrm{orb}_G(x)$, is the set $\{y \in X | \exists g \in G$ such that $g.x = y\}$. We recall that the *stabilizer of* x, denoted $\mathrm{stab}_G(x)$ is the set $\{g \in G | g.x = x\}$, it is always a subgroup.

We also recall some quick facts about group actions so suppose G acts on X.

(1) X is the disjoint union of its distinct orbits.

(2) $|\mathrm{orb}_G(x)| = [G : \mathrm{stab}_G(x)]$ (the size of an orbit is the same as the number cosets of the stabilizer).

(3) If G is finite, then (2) reduces to $|\mathrm{orb}_G(x)| \cdot |\mathrm{stab}_G(x)| = |G|$.

1. Let G be a group and H be a subgroup. Set X to be the set of *LEFT* cosets of H (notice that X is not necessarily a group because H is not normal). Prove that G acts on X with the following action.

$$g.(aH) = (ga)H.$$

Solution: First we show that the action is well defined so suppose that $cH = c'H$. Then $g.(cH) = (gc)H = g(cH) = g(c'H) = (gc')H = g.(c'H)$. Now we prove that the two properties of an action, (i) and (ii) above hold. For (i), simply notice that $e.(cH) = (ec)H = cH$. For (ii) observe that $a.(b.(cH)) = a.((bc)H) = (a(bc))H = ((ab)c)H = (ab).(cH)$ as desired.

2. With the notation as in **1.**, consider $G = S_3$ and $H = \langle(12)\rangle$. Compute the orbits and stabilizers of all the elements of X.

Solution: First we write down X, the left cosets of H. Note

$$X = \{\{e, (12)\}, \{(13), (123)\}, \{(23), (132)\}\}.$$

We first compute the orbit of $\{e, (12)\}$. Notice that $e.\{e, (12)\} = \{e, (12)\}$, $(13).\{e, (12)\} = \{(13), (123)\}$ and $(23).\{e, (12)\} = \{(23), (132)\}$. Thus $\mathrm{orb}_{S_3}(\{e, (12)\}) = X$. Since the orbits are disjoint, this is the only orbit.

Now we compute the stabilizer of each element.

$$\mathrm{stab}_{S_3}(\{e, (12)\}) = \{e, (12)\}$$
$$\mathrm{stab}_{S_3}(\{e, (13)\}) = \{e, (23)\}$$
$$\mathrm{stab}_{S_3}(\{e, (23)\}) = \{e, (13)\}$$

3. Fix the notation as in **1.**, and fix $g \in G$. We define the function $\tau_g : X \to X$ by the rule.

$$\tau_g(aH) = (ga)H$$

Prove that τ_g really is a permutation (ie, prove it is bijective).

Solution: Consider the function $\tau_{g^{-1}}$. I claim that $\tau_g \circ \tau_{g^{-1}} = \mathrm{id}_X = \tau_{g^{-1}} \circ \tau_g$.

First notice that for any $aH \in X$, $\tau_g \circ \tau_{g^{-1}}(aH) = \tau_g((g^{-1}a)H) = (g(g^{-1}a))H = aH$. Likewise $\tau_{g^{-1}} \circ \tau_g(aH) = \tau_{g^{-1}}((ga)H) = (g^{-1}(ga))H = aH$. This proves τ_g has an inverse function and thus τ_g is a permutation.

4. Fix the notation as in **1.**. Recall that S_X is the set of permutations on X prove that $\phi : G \to S_X$ defined by the rule $\phi(g) = \tau_g$ is group homomorphism. Also prove that the kernel of ϕ is contained within H.

Solution: First notice that $\tau_{gg'}(aH) = ((gg')aH) = (g(g'a))H = \tau_g((g'a)H) = \tau_g \circ \tau_{g'}(aH)$. Since aH was arbitrary, this proves that $\tau_{gg'} = \tau_g \circ \tau_{g'}$. Therefore $\phi(gg') = \tau_{gg'} = \tau_g \circ \tau_{g'} = \phi(g) \circ \phi(g')$ which proves that ϕ is a homomorphism.

Suppose now that $g \in \ker\phi$. Thus $\phi(g) = \tau_g$ is the identity. In other words, $\tau_g(aH) = aH$ for all $aH \in X$. In particular, $\tau_g(eH) = gH = H$. Therefore $g \in H$ and so $\ker\phi \subseteq H$ as desired.