WORKSHEET # 4 SOLUTIONS

MATH 435 SPRING 2011

We first recall some facts and definitions about cosets. For the following facts, G is a group and H is a subgroup.

(i) For all $g \in G$, there exists a coset aH of H such that $g \in aH$. (One may take $a = g$).
(ii) Cosets are equal or are disjoint. In other words, if $aH \cap bH \neq \emptyset$, then $aH = bH$.
(iii) Properties (i) and (ii) may be summarized by saying: "The (left) cosets of a subgroup partition the group."
(iv) If H is finite, then $|H| = |aH|$ for every coset aH of H (this holds for infinite cosets too).
(v) Cosets of H are generally *NOT* subgroups themselves.
(vi) Two cosets aH and bH are equal if and only if $b^{-1}a \in H$.
(vii) The subgroup H is called *normal* if $aH = Ha$ (in other words, if the left and right cosets of H coincide, this does not mean $ah = ha$ for all $h \in H$, but it does mean that for all $h \in H$, there exists another $h' \in H$ such that $ah = h'a$).

1. Consider the group $G = \mathbb{Z}$ under addition with subgroup $H = 4\mathbb{Z}$. Write down the four cosets of H.

Solution: The cosets are

$$\begin{aligned} 0 + H &= \{\cdots - 8, -4, 0, 4, 8, 12, \dots\} \\ 1 + H &= \{\cdots - 7, -3, 1, 5, 9, 13, \dots\} \\ 2 + H &= \{\cdots - 6, -2, 2, 6, 10, 14, \dots\} \\ 3 + H &= \{\cdots - 5, -1, 3, 7, 11, 15, \dots\} \end{aligned}$$

2. With the same setup as the first problem, consider the cosets $1 + H$ and $2 + H$. If you add these two cosets together, what do you get? Write down a general formula for the sum of $n+H$ and $m+H$.

Solution: Adding the first two cosets I get:

$$(1 + H) + (2 + H) = \{\cdots - 7, -3, 1, 5, 9, 13, \dots\} + \{\cdots - 6, -2, 2, 6, 10, 14, \dots\}$$

All possible sums from those two sets equals $\{\cdots - 5, -1, 3, 7, 11, 15, \dots\} = 3 + H$. In general, we have $(n + H) + (m + H) = (n + m) + H$, which can also be written as $(n + m \mod 4) + H$.

3. Prove that for any integer n, the cosets of $n\mathbb{Z} \subseteq \mathbb{Z}$ form a cyclic group under addition.

Solution: The cosets of $H := n\mathbb{Z}$ in $\mathbb{Z}$ are just $0 + H, 1 + H, \dots, (n - 1) + H$. Based on the type of computation done above, the summation $(a + H) + (b + H) = (a + b \mod n) + H$ is a binary operation, the associativity follows from the associativity of arithmetic mod n. Certainly $0 + H$ is the identity, $a + H$ has inverse $-a + H$ and it's easy to see that $1 + H$ is a generator, and thus the group is cyclic.

At some level what I've written above is not a complete solution. However, you should carefully verify (and read in the book) about the details not mentioned here.

4. Suppose that G is a group and H is a *normal* subgroup (but do not assume that G is Abelian). We will show that the set of cosets of H form a group under the following operation.

$$(aH)(bH) = (ab)H.$$

First however, we need to prove that this is well defined. Suppose that $a'H = aH$ and $b'H = bH$. Prove that

$$(ab)H = (a'b')H.$$

Solution: Proving that the last displayed equation holds will prove that the operation is well defined. We will show $(ab)H \subseteq (a'b')H$, the other inclusion will follow by symmetry.

Choose an element $abh \in (ab)H$ (where $h \in H$). Choose an element $h_1 \in H$ such that $abh = ah_1b$. We know that $aH = a'H$ so there exists $h_2 \in H$ such that $ah_1 = a'h_2$. Thus $abh = ah_1b = a'h_2b$. Again, because H is normal, this equals $a'bh_3$ and finally because $bH = b'H$, there exists $h' \in H$ such that $a'bh_3 = a'b'h' \in (a'b')H$ as desired.

Notice I didn't worry about the parentheses / associativity, but we are working in a group and so this is harmless.

4. Prove that the operation above indeed forms a group. The set of cosets of H with the group operation below is denoted G/H. It is called the *quotient group of* G *modulu* H or simply G *mod* H.

Solution: Now that we know the operation is well defined, we prove it forms a group.

(1) For associativity, notice that

$$((aH)(bH))(cH) = ((ab)H)(cH) = ((ab)c)H = (a(bc)H = (aH)((bc)H) = (aH)((bH)(cH)).$$

(2) For identity, notice that $(eH)(aH) = aH = (aH)(eH)$.

(3) For inverses, notice that $(a^{-1}H)(aH) = (a^{-1}a)H = eH = (aa^{-1}H) = (aH)(a^{-1}H)$ as desired.

5. Show that there is a surjective group homomorphism $G \to G/H$ whose kernel is exactly H.

Solution: Consider the function $\phi : G \to G/H$ defined by the rule $\phi(g) = gH$. This function is certainly well defined (ask yourself why). $\phi(ab) = (ab)H = (aH)(bH) = \phi(a)\phi(b)$ and indeed is thus a group homomorphism. It is certainly surjective because for any coset aH, $\phi(a) = aH$.

To analyze the kernel, suppose that $\phi(a)$ is the identity of G/H, in other words, suppose that $aH = eH$. But that is equivalent to $a = e^{-1}a \in H$ by property (vi) on the first page. In other words, $\phi(a) = e_{G/H}$ if and only if $a \in H$.

6. Find an example of a group G and a normal subgroup H such that both G and H are non-Abelian but G/H is Abelian.

Solution: Consider $G = S_4$ and $H = A_4$. Both G and H are not Abelian. However, G/H has 2 elements in it. Because 2 is prime, G/H is cyclic and so G/H is Abelian.

By the way, the easiest answer is to choose G to be any non-Abelian group and then set $H = G$.