WORKSHEET # 2

MATH 435 SPRING 2011

Definition 0.1. A permutation $\alpha \in S_n$ is called *even* if it can be written as a product of an even number of transpositions (ie, cycles of the form (ij)). A permutation $\alpha \in S_n$ is called *odd* if it isn't even.

1. Set A_n to be the set of all even permutations in S_n. Prove that A_n is a group with binary operation composition (ie, the induced binary operation from S_n).

Solution: First we prove that composition is a binary operation: If α can be written as a product of an even number n of 2-cycles, and β can also be written as a product of an even number m of 2-cylces, then $\alpha\beta$ can be written as a product of $n + m$, which is even, 2-cycles. Thus composition is indeed a binary operation.

Now we prove that A_n is indeed a group. Associativity is immediate because function composition is always associative. The identity $e = (12)(12)$ can certainly be written as an even number of two cycles, thus $e \in A_n$. For inverses, suppose that $\alpha = (ab)(cd)\dots(wx)$ where there are an even number of pairs transpositions. $\alpha^{-1} = (wx)\dots(ab)$ thus can also be written as an even number of transpositions. Thus A_n is indeed a group.

2. Identify all the elements of A_2, A_3 and A_4. Are any of these groups Abelian?

Solution:

(i) A_2. In this case $S_2 = \{e, (12)\}$ and so $A_2 = \{e\}$. This group is certainly Abelian (there is nothing to check).

(ii) A_3. Now $S_3 = \{e, (12), (13), (23), (123), (132)\}$. Thus $A_3 = \{e, (123) = (13)(12), (132) = (12)(13)\}$. This group is also Abelian since $(123)(132) = e = (132)(123)$ (note for any α, $\alpha e = \alpha = e\alpha$, likewise $\alpha\alpha = \alpha^2 = \alpha\alpha$ – in this last case, the order of α multiplied by itself certainly doesn't matter).

(iii) A_4. I won't write down S_4, but I will note that any n-cycle is even if and only if $n - 1$ is even. Note that $(12\dots n) = (1n)\dots(12)$ which has $n-1$ terms in its product. Thus, $A_4 = \{e, (12)(34), (13)(24), (14)(23), (123), (132), (124), (142), (134), (143), (234), (243)\}$. This group is not Abelian since $(123)(124) = (13)(24)$ but $(124)(123) = (14)(23)$.

3. Conjecture and prove a formula for the number of elements in A_n
Hint: Compare the size of A_2, A_3 and A_4 with the size of S_2, S_3 and S_4 respectively. To prove your formula, consider the function from the set of even permutations to the set of odd permutations given by multiplication (on the left) by (12) and show it is bijective.

Solution: We first make the assumption that $n \geq 2$, as in the case that $n = 1$, our proposed formula breaks down (in this case $A_n = S_n = \{e\}$). Our formula is $n!/2$ since $A_2 = 1$ while $S_2 = 2 = 2!$, and $A_3 = 3$ while $S_3 = 6 = 3!$ and $A_4 = 12$ while $S_4 = 24 = 4!$. We now prove that this formula is correct.

Let $B_n = S_n \setminus A_n$. It is sufficient to show that B_n (the set of odd permutations) is the same size as A_n because then the number of elements of A_n is the number of elements of S_n over 2, or $n!/2$.

Consider the function $\phi : A_n \to B_n$ defined by the rule $\phi(\alpha) = (12)\alpha$. We will show that ϕ is bijective proving the theorem.

For injectivity, suppose first that $\phi(\alpha) = \phi(\beta)$, thus $(12)\alpha = (12)\beta$ and so $\alpha = (12)(12)\alpha = (12)(12)\beta = \beta$ which proves that ϕ is injective.

For surjectivity, choose now $\gamma \in B_n$, γ is an odd permutation and so $(12)\gamma$ is even. But now $\phi((12)\gamma) = (12)(12)\gamma = \gamma$ and so ϕ is indeed surjective.

Thus ϕ is bijective and the proof is completed.

4. Show that a permutation with odd order must always be an even permutation.
Solution: Suppose that $\alpha^{2n+1} = e$ for some integer n. Writing α as a product of m transpositions, and plugging this into α^n, we see that a product of $m(2n+1)$ transpositions is equal to e. But in class we showed that e can only be written as a product of an even number of transpositions. Thus $m(2n+1)$ is even and thus m is also even, which proves that α is an even permutation as desired.