HOMEWORK # 9
DUE WEDNESDAY MARCH 30TH

MATH 435 SPRING 2011

In this homework assignment, all rings with be commutative associative with unity (multiplicative identity). Ring homomorphisms will always be assumed to send 1 to 1. The terms that appear on this homework assignment *WILL* appear on the exam.

1. An ideal I is called *radical* if for every $x \in R$ such that $x^n \in I$ (for some n), then $x \in I$ also. Prove that I is radical if and only if R/I has no nilpotent elements.
2. Suppose that $x \in R$ is a nilpotent element. Prove that $1 + x$ is a unit.
3. Suppose that I and J are ideals of a ring. We define $IJ = \{x \in R | x$ is equal to a finite sum of ij for some $i \in I, j \in J\}$. Show that both IJ and $I \cap J$ are ideals of R. Further show that $IJ \subseteq I \cap J$.
4. A ring R is called a *principal ideal domain* if it is an integral domain and every ideal $I \subseteq R$ is principal, in other words $I = (r)$ for some R in R. Show that $\mathbb{Z}[i]$ is a principal ideal domain.
5. Consider the ring R of continuous functions $\phi : \mathbb{R} \to \mathbb{R}$. Prove that the subset $I = \{f \in R | f(1) = 0\}$ is an ideal but that $A = \{f \in R | f(1) \in \mathbb{Z}\}$ is not an ideal.
6. Suppose that A is an integral domain of positive characteristic. Suppose that $I \subsetneq A$ is an ideal. Prove that A/I has the same characteristic as A. However, give an example which demonstrates that A/I need not be an integral domain.
7. Suppose that F is a field with finitely many elements but not $\mathbb{Z}_{\text{ mod } 2}$. Prove that the sum of all the elements in the field is equal to zero.
8. Find the characteristic of $\mathbb{Z}[i]/(2 + i)$.
9. Give an example of a prime ideal which is not maximal. An ideal $I \subseteq R$ is called *maximal* if there are no proper ideals J such that $I \subsetneq J \subsetneq R$.
10. Show that the ideal $(2 + \sqrt{2}) \subseteq \mathbb{Z}[\sqrt{2}]$ is not prime.
11. Suppose that S is a ring. Show that there is a unique ring homomorphism $\phi : \mathbb{Z} \to S$ which sends 1 to 1. In particular, conclude that there is only one ring homomorphism $\mathbb{Q} \to \mathbb{Q}$ which sends 1 to 1.
12. Fix a field k and consider the subring $k[x^2, x^3] \subseteq k[x]$ (the former ring is all polynomials with zero coefficient in front of x). Prove that $k[x^2, x^3]$ is not a principal ideal domain.
13. Suppose that $\phi : R \to S$ is a surjective ring homomorphism. Prove that if R is a principal ideal domain the every ideal in S is also principal. However, give an example to show that S need not be an integral domain.
14. Let $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}$ and $\mathbb{Q}[\sqrt{5}] = \{a + b\sqrt{5} | a, b \in \mathbb{Q}\}$ Prove that both rings are fields but that they are not isomorphic.