HOMEWORK # 7
DUE FRIDAY MARCH 4TH

MATH 435 SPRING 2011

1. Fix p a prime. Suppose that G is a finite group in which every element's order is a power of p. Prove that $|G| = p^n$ for some integer n.
2. Suppose that $|G| = p^n$ and further suppose that G is simple. Prove that $|G| = p$.
3. Suppose that G is a group with $|G| = mp$ where p is a prime and $1 < m < p$. Prove that G is not simple.
4. Suppose that G is a group acting on X and that $\phi : A \to G$ is a group homomorphism. Prove that a acts on X by the rule $a.x = \phi(a).x$.
5. Show that the additive group $\mathbb{R}$ acts on $\mathbb{R}^2$ by the rule $r.(x, y) = (x + ry, y)$.
6. Consider the group G of all matrices

$$\left\{ \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \mid \theta \in \mathbb{R} \right\}$$

These are matrices which rotate by θ-degrees clockwise. We view these matrices acting on $\mathbb{R}^2$ by multiplication. Describe the orbits of this action. For each point $\mathfrak{q} \in \mathbb{R}^2$, describe the stabilizer of $\mathfrak{q}$.
7. Find a finite group G acting on the set $\{1, 2, 3\}$ such that the stabilizer of 1 is bigger than the stabilizer of 2, which is the same size as the stabilizer of 3.
8. Given a subgroup $A \subseteq G$, consider the set $N_G(A) = \{g \in G | gAg^{-1} = A\}$. This subset is called the normalizer of A. Prove that A is normal if and only if $N_G(A) = G$.
9. Prove that $N_G(A)$ is always a subgroup of G which contains A. Further prove it is the unique largest subgroup in which A is normal.
10. Find an example of a subgroup $A \subseteq G$ such that $N_G(A) \supsetneq A$ but that $N_G(A) \subsetneq G$.
11. Give an example of a cyclic group acting on the set $\{1, 2, 3, 4\}$ such that each orbit has two elements in it.
Extra Credit #1: Prove the extra credit problem from the exam. 2 points to your homework score. Due Wednesday March 2nd.
Hint: Set $M = B/A$ (note that $A \to B$ is injective, so viewing A as a subgroup of B is essentially harmless).

Extra Credit #2: Study the Chinese remainder theorem as follows. Worth up to 4 points on your homework score. Due Monday March 14th. The most basic case of the Chinese remainder theorem is as follows. Suppose that n_1 and n_2 are relatively prime integers and a_1, a_2 are any pair of integers. Then the system of equations $x \equiv_{\mod n_1} a_1$, $x \equiv_{\mod n_2} a_2$ has a solution.

Consider the following generalization. Suppose that G is a group and N_1 and N_2 are normal subgroups such that $N_1N_2 = G$ (explain why this is related to relative primality of n_1 and n_2). Fix two elements $g_1, g_2 \in G$.

Question 0.1. Does the system of equations (in cosets) $xN_1 = g_1N_1$ and $xN_2 = g_2N_2$ always have a solution for $x \in G$? Either prove it or provide a counter-example. If it doesn't hold, does it hold if you assume G is Abelian? What if you assume G is cyclic.

The more general Chinese remainder theorem holds for collections of integers $n_1, \dots, n_k$ which are pairwise relatively prime, and for arbitrary collections of integers $a_1, \dots, a_k$. In particular, the system of equations (for all i) $x \equiv_{\mod n_i} a_i$ always has a solution.

Question 0.2. Suppose that G is a group and $N_1, \dots, N_k$ are normal subgroups such that $N_iN_j = G$ for all $i \neq j$. Fix elements $g_1, \dots, g_k \in G$. Does the system of equations $xN_1 = g_1N_1, \dots, xN_k = g_kN_k$ always have a solution for some $x \in G$? What if G is Abelian, or cyclic? Prove it or give a counter-example.