HOMEWORK # 6
DUE FRIDAY FEB. 18TH

MATH 435 SPRING 2011

1. Suppose that G is a finite group. Prove that there is an injective group homomorphism $G \to S_n$ for some integer $n \geq 1$. This fact is sometimes called: "every group is isomorphic to a subgroup of a permutation group."
2. Prove that $U(8)$ is isomorphic to $U(12)$.
3. Prove that $\mathbb{Z}$ and $\mathbb{Q}$ are not isomorphic (as groups under addition).
4. Consider the group $G = \mathbb{C} \setminus \{0\}$ of non-zero complex numbers under multiplication. Consider the function $\phi : G \to G$ which sends $a + bi$ to $a - bi$. Prove that this gives an isomorphism of G with itself.
5. Consider the group $U(30)$ and the subgroup $H = \langle 7 \rangle$. Compute all the cosets of H.
6. Find all the normal subgroups of S_4.
7. Consider the group $GL(2)$ of invertible 2×2-matrices. Set H to be the subgroup of $GL(2)$ to be the group of matrices with determinant equal to 1. Prove that H is a normal subgroup of $GL(2)$.
8. Consider $H \subseteq U(30)$ in 5. Is $U(30)/H$ abelian? Is it cyclic?
9. Set G to be a group and set $Z(G) = \{x \in G | xg = gx \forall g \in G\}$. Show that $Z(G)$ is always a normal subgroup of G.
10. Suppose that G is a group and $G/Z(G)$ is cyclic. Prove that $Z(G) = G$.
11. Suppose that G is a group with $|G| = p^3$ where p is prime. Prove that $|Z(G)| \neq p^2$.
12. Suppose that N is a normal subgroup of a group G and H is any other subgroup. Show that

$$NH = \{nh | n \in N, h \in H\}$$

is a subgroup of G. If H is normal, prove that NH is also normal. Give an example where N and H are not normal and that NH is not even a subgroup.