HOMEWORK # 5
DUE FRIDAY FEB. 11TH

MATH 435 SPRING 2011

1. Prove that A_4 has no subgroup of order 6.
2. For each natural number $n \in \mathbb{N}$, find an *infinte* group G and a subgroup $H \subseteq G$ such that $[G : H] = n$.
3. Let G be a finite group with subgroups $H \subseteq K$. Prove that

$$[G : H] = [G : K][K : H].$$

4. Consider $\mathbb{Q}$, the group of rational numbers under addition. Prove that there is no proper subgroup $H \subseteq \mathbb{Q}$ such that $[\mathbb{Q} : H] < \infty$.
5. We define $\phi(n)$ to be the number of positive integers less than n and relatively prime to n (this is called *Euler's phi function* or *the totient function*). Show that if $a, n \in \mathbb{Z}$ are such that $\gcd(a, n) = 1$, then

$$a^{\phi(n)} \equiv_{\text{ mod } n} 1.$$

6. Suppose that G is a group with more than one element and that G has no proper non-trivial subgroups. Prove that $|G|$ is both finite and prime.
7. Suppose that G is a group and that H is a subgroup. Prove that the map $\psi : H \to G$ defined by the rule $\psi(h) = h$ is an injective group homomorphism.
8. Suppose that A and B are groups and that $A = \langle a \rangle$ is cyclic. Suppose that $\phi : A \to B$ is a group homomorphism. Prove that both the image of ϕ and the kernel of ϕ are cyclic.
9. Suppose that $\phi : A \to B$ is a surjective group homomorphism and that A is Abelian. Prove that B is Abelian. However, give an example showing that the converse is false.
10. Suppose that G is a group. Fix an element $x \in G$ and consider the function $\phi : G \to G$ defined by the rule $\phi(g) = xg$. Prove that ϕ is never a group homomorphism unless $x = e$.
11. Suppose that G is a group. Consider the set of bijective group homomorphisms $\phi : G \to G$. This set is called *the Automorphisms of* G and is denoted by $\text{Aut}(G)$. Prove that $\text{Aut}(G)$ is a group under composition.