HOMEWORK # 12
DUE WEDNESDAY APRIL 27TH

In this homework assignment, all rings with be commutative associative with unity (multiplicative identity). Ring homomorphisms will always be assumed to send 1 to 1.

1. Using only the fact that $\mathbb{R}$ is uncountable, prove that there must exist a transcendental element $\alpha \in \mathbb{R}$ (this is much easier than proving that π or e is transcendental).
Hint: Recall that a set S is called uncountable if it is infinite and there is no bijection $\phi : S \to \mathbb{Z}$. You may use the fact that $\mathbb{Q}$ is countable, a countable (or finite) union of countable sets is still countable, and finite product of countable sets is also countable.
2. Prove that every algebraic extension of $\mathbb{R}$ is degree 1 or 2.
Hint: You may take as given that $\mathbb{C}$ is algebraically closed, meaning that there is no finite extension of $\mathbb{C}$.
3. Find the splitting field of $x^5 - 2$ over $\mathbb{Q}$. Find the degree of this splitting field over $\mathbb{Q}$.
4. Suppose that $\mathbb{F}$ is a finite field. Prove that the group of units, $G = \mathbb{F} \setminus \{0\}$, is a cyclic group. Do *not* use the characterization of a cyclic group in Rotman which says that a group is cyclic if and only it has exactly one subgroup of order m for each integer m dividing the order of the group.
5. Find an element $\alpha \in \mathbb{R}$ such that $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\alpha)$.
6. Suppose that k is a field and K is an extension field. Further suppose that K_1 and K_2 are two extensions of k contained in K. We then define K_1K_2 to be the smallest subfield of K containing both K_1 and K_2. Prove that

$$[K_1K_2 : k] \leq [K_1 : k][K_2 : k]$$

Further, give an example which shows that this inequality can sometimes be $=$. Give another example which shows that this inequality can sometimes be $<$.
7. Prove that d divides n if and only if $x^d - 1$ divides $x^n - 1$.
8. Find $[\mathbb{F}_{p^n} : \mathbb{F}_{p^d}]$ if d divides n (in particular, show that $\mathbb{F}_{p^d}$ can be viewed as a subfield of $\mathbb{F}_{p^n}$).
9. Find $\mathrm{Aut}(L/\mathbb{Q})$ where L is the splitting field you constructed in problem **3.**.
10. Find the smallest field with exactly 6 subfields, note $\{0\}$ is not a subfield.
11. Prove or disprove. Fix k to be a field of characteristic $p > 0$, the ring homomorphism $\phi : k \to k$ defined by $\phi(x) = x^p$ is always:

(a) the identity.
(b) an isomorphism.
(c) an injection.

12. Prove that $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}, {}^{1/3}\sqrt{2}, {}^{1/4}\sqrt{2}, \dots)$ is an algebraic extension of $\mathbb{Q}$ but not a finite extension of $\mathbb{Q}$.

Extra Credit (Due Friday, April 29th) *Do NOT turn in with your regular homework.*

1. Suppose that k is an algebraically closed field. Prove that the maximal ideals of $k[x, y, z]$ are in bijection with the vector space k^3. For each maximal ideal $\mathfrak{m} \subseteq k[x, y, z]$, and each $f(x, y, z) \in k[x, y, z]$ explain why the coset $f(x, y, z) + \mathfrak{m}$ is *evaluation at the point in k^3 corresponding to* $\mathfrak{m}$. (1 point)
2. Suppose that R is an integral domain. A *multiplicative system of* R is a collection W of non-zero elements of R, also containing 1, such that for all $x, y \in W$, $xy \in W$. Given $f \in R$, show that $\{1, f, f^2, f^3, \dots\}$ is multiplicative system. Further show that for any ideal $P \subseteq R$, $R \setminus P$ is a multiplicative system if and only if P is prime. (1 point)
3. For any multiplicative system $W \subseteq R$, consider the subset $W^{-1}R = \{[(a, w)] \in K(R) | a \in R, w \in W\}$. Show that $W^{-1}R$ is a subring of $K(R)$ which contains R. (1 point)
4. Suppose that R is an integral domain and $W \subseteq R$ is a multiplicative system. Prove that the prime ideals of $W^{-1}R$ are in bijective correspondence with the prime ideals Q of R such that $Q \cap W = \emptyset$. Furthermore, if P is a prime ideal of R and $W = R \setminus P$, prove that $W^{-1}R$ has a unique maximal ideal. (1 point)
5. Set $R = k[x, y, z]$ and suppose $f \in R$. Set $W = \{1, f, f^2, f^3, \dots\}$. Describe geometrically the maximal ideals of $W^{-1}R$ as a subset of k^3 (using a similar idea to problem **4.**). (1 point)