HOMEWORK # 11
DUE WEDNESDAY APRIL 20TH

MATH 435 SPRING 2011

For this homework, we assume all rings are commutative, associative and with multiplicative identity. We assume that all homomorphisms send 1 to 1.

1. Suppose k is a field and $f(x)$ and $g(x)$ are elements in $k[x]$ such that $(f(x), g(x)) = k[x]$. Prove
$$k[x]/(f(x) \cdot g(x)) \cong k[x]/(f(x)) \oplus k[x]/(g(x)).$$
2. Prove that for every integer $d \geq 1$, there is an irreducible polynomial of degree d in $\mathbb{Q}[x]$.

3. Find a polynomial $p(x) \in \mathbb{Q}[x]$ such that $\mathbb{Q}[x]/(p(x)) \cong \mathbb{Q}(\sqrt{1+\sqrt{3}})$. Justify your result.

4. Let $\mathbb{F}_2 = \mathbb{Z}_{\text{mod}2}$. Consider $f(x) = x^3 + x + 1 \in \mathbb{F}_2[x]$. Further suppose that E is some extension field of $\mathbb{F}_2$ and that $a \in E$ is such that $f(a) = 0$. Write down all of the elements of $\mathbb{F}_2(a)$ and write down a complete multiplication table for $\mathbb{F}_2(a)$ as well.

5. Find the splitting field E for $x^4 - 1$ over $\mathbb{Q}$. Find an element $a \in \mathbb{C}$ such that $\mathbb{Q}(a) = E$.

6. Consider $f(x) = \sum_{i=0}^{n} a_i x^i \in k[x]$. As in class on Friday (hopefully), we considered $f'(x) = \sum_{i=1}^{n} i a_i x^{i-1} \in k[x]$. Prove that $(f(x) + g(x))' = f'(x) + g'(x)$. Also prove that $((f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$.

Hint: For the second part, use induction on the degree of $f(x)$.

7. Show that $\mathbb{Q}(4 - i) = \mathbb{Q}(1 + i)$.

8. Prove or disprove that $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{-3})$ are isomorphic as fields.

9. Suppose that K is a field and $k \subseteq K$ is a subfield. Consider the set G of ring isomorphisms $\phi : K \to K$ such that $\phi(x) = x$ for all $x \in k$. Prove that G is a group, it is usually denoted by $\text{Aut}(K/k)$. Compute $\text{Aut}(\mathbb{Q}(\sqrt{2}), \mathbb{Q})$.

Note: We use the following setup for the rest of the homework assignment. Suppose that R is an integral domain. Consider the set $L(R) = R \times (R \setminus \{0\})$ (ordered pairs where the second element can't be zero). We declare that two elements $(a, b), (a', b') \in L(R)$ to be equivalent if $ab' = a'b$. Now set $K(R)$ to be the set of equivalence classes of $L(R)$. We will define multiplication and addition on $K(R)$ as follows. For $[(a, b)], [(a', b')] \in K(R)$ we define
$$[(a, b)] + [(a', b')] = [(ab' + a'b, bb')]$$
(the purpose of the square brackets $[(a, b)]$ is to remind us that elements are equivalence classes). We define $[(a, b)] \cdot [(a', b')] = [(aa', bb')]$.

10. Prove that the multiplication and addition rules above are well defined, and that they satisfy the axioms of a commutative ring.

11. Prove that $K(R)$ is a field.

12. Prove that there is a ring homomorphism $\phi : R \to K(R)$ defined by $\phi(r) = [(r, 1)]$. Show that this ring homomorphism is always injective and further show that it is bijective, in other words an isomorphism, if and only if R is a field.

13. Show that $K(\mathbb{Z})$ is isomorphic to $\mathbb{Q}$ via the isomorphism $\phi([a, b]) = a/b$, which you must show is well defined. Because of this, for any integral domain R, the elements $[(a, b)] \in K(R)$ are always written down as a/b. Demonstrate that the addition and multiplication rules for fractions are exactly the rules we wrote down above.