# HOMEWORK # 10
# DUE WEDNESDAY APRIL 13TH

MATH 435 SPRING 2011

For this homework, we assume all rings are commutative, associative and with multiplicative identity. We assume that all homomorphisms send 1 to 1.

**1.** Show that a ring $R$ is Noetherian if and only if for every ideal $I \subseteq R$, there exists elements $x_1, \ldots, x_n \in I$ such that $I = (x_1, \ldots, x_n)$. We talked through this in class on Wednesday April 6th, but now write down the details carefully.

**2.** Suppose that $R$ is a unique factorization domain. Prove that every irreducible element in $R$ is prime.

**3.** Prove or disprove, every subring of a PID is a PID.

**4.** Same question as #**3.**, but with UFD.

**5.** Show that $1 - i$ is irreducible in $\mathbb{Z}[i]$.

**6.** Prove that every non-zero prime ideal in a PID is maximal.

**7.** Suppose that $\phi : R \to S$ is a surjective ring homomorphism. Suppose that $x \in R$ is an irreducible element. Is it true that $\phi(x)$ is also irreducible? Prove it or give a counter-example.

**8.** Show that a non-constant polynomial from $\mathbb{Z}[x]$ that is irreducible (as an element of $\mathbb{Z}[x]$) is primitive.

**9.** Show that $x^4 + 1 \in \mathbb{Q}[x]$ is irreducible. But $x^4 + 1 \in \mathbb{R}[x]$ is reducible (not irreducible).

**10.** Explicitly construct a field with 49 elements.

**11.** Determine which of the following polynomials below are irreducible over $\mathbb{Q}$ (ie, irreducible elements of $\mathbb{Q}[x]$.

(a) $x^5 + 9x^4 + 12x^2 + 6$
(b) $x^4 + x + 1$
(c) $x^4 + 3x^2 + 3$
(d) $x^5 + 5x^2 + 1$
(e) $\frac{5}{2}x^5 + \frac{9}{2}x^4 + 15x^3 + \frac{3}{7}x^2 + 6x + \frac{3}{14}$