Warm-up homework set: Due Wednesday, January 12th

(1) What is wrong with the following inductive proof that either all lightbulbs are all off or lightbulbs are all on?

> "We will show that for any set of n lightbulbs, all lightbulbs in the set are all on *or* all lightbulbs in the set are all off. For the base case, we consider a set with 1 lightbulb in it. Clearly that lightbulb is either off or on which proves the base case. Now suppose that S is a set of $n+1$ lightbulbs (we label them 0 to n). Suppose that $A \subset S$ is the set of lightbulbs labelled 0 to $n-1$ and $B \subset S$ is the set of lightbulbs labelled 1 to n. By our inductive hypothesis, all the lightbulbs in the set A are either all on or all off, likewise with the set B. Now, notice that if A is "on/off", then so is B, because A and B have the lightbulbs $1, \ldots, n-1 = A \cap B$ in common. Thus all the lightbulbs in $S = A \cup B$ are also either all on or all off."

(2) Find a formula for $1 + 3 + 5 + \cdots + (2n-1)$ and use inductive reasoning to prove that your formula is correct.

(3) Suppose that $f : S \to T$ and $g : T \to U$ are two functions and consider the composition $g \circ f : S \to U$.
   (a) Suppose that $g \circ f$ is injective, prove that f is also injective.
   (b) Suppose that g and f are both surjective, prove that $g \circ f$ is also surjective.
   (c) Give an example of two functions g and f such that g is surjective, f is not surjective, but $g \circ f$ is surjective.

(4) Consider the following proof that there are infinitely many prime natural numbers.

> "Suppose that there were finitely many primes, $p_1, \ldots, p_n$. Consider the new number $m = p_1 \cdot p_2 \ldots p_n + 1$. It is clear that $m > p_i$ for $i = 1, \ldots, n$ and so m is not prime. But $p_i \not| m$ for each $i = 1, \ldots, n$ since $m = p_i(\prod_{j \neq i} p_i) + 1$. Now, every integer m is a product of primes by the fundamental theorem of arithmetic, but no prime divides m, a contradiction.

The proof is correct, but consider the following question inspired by it. If we set $p_1, \ldots, p_n$ to be the first n primes, and define $m = p_1 \cdot p_2 \ldots p_n + 1$, is it true that m is always a prime number?

Either prove that this is correct or provide a counter-example.

(5) Prove that the square root of 12 is irrational
Hint: Suppose $\sqrt{12} = a/b$ for some positive $a, b \in \mathbb{Z}$, square both sides and derive a contradiction using unique factorization.

(6) Show that $\gcd(a, bc) = 1$ if and only if $\gcd(a, b) = 1$ and $\gcd(a, c) = 1$.