MATH 435, EXAM #1

Your Name

- You have 50 minutes to do this exam.
- No calculators!
- No notes!
- For proofs/justifications, please use complete sentences and make sure to explain any steps which are questionable.
- Good luck!

Problem	Total Points	Score
1	30	
2	25	
3	20	
4	25	
EC	10	
Total	100	

1. Definitions and short answers.
(a) What is the definition of a group homomorphism? (5 points)

Solution: Given two groups A and B, a group homomorphism is a function $\phi : A \to B$ such that for all $a, a' \in A$, we have $\phi(aa') = \phi(a)\phi(a')$.

(b) Give an example of a subgroup H in a group G such that H is not normal in G. (5 points)

Solution: Set $G = S_3$ and $H = \langle (12) \rangle$.

(c) Write $(123)(234)(345) \in S_5$ in disjoint cycle notation. (5 points)

Solution: $(12)(45)$.

(d) Give an example of an injective but not surjective group homomorphism. (5 points)

Solution: $\phi : 2\mathbb{Z} \to \mathbb{Z}$ defined by the rule $\phi(x) = x$.

(e) Justify the statement, *every group S_n has a normal subgroup.* (5 points)

Solution: The subgroups $\{e\}, S_n$ and A_n are all normal in S_n.

(f) State Lagrange's theorem. (5 points)

Solution: Suppose that G is a finite group, then for any subgroup $H \subseteq G$, the order of H divides the order of G.

2. Consider the group $U(20)$ and the cyclic subgroup $H = \langle 9 \rangle$.
(a) Write down all the cosets of H (5 points)

Solution: First note that $H = \{9, 1\}$ and $U(20) = \{1, 3, 7, 9, 11, 13, 17, 19\}$. Therefore the cosets of H are $\{9, 1\}$, $\{3, 7\}$, $\{19, 11\}$, $\{13, 17\}$.

(b) Write down the multiplication table for the group $U(20)/H$. (12 points)

Solution:

	{9, 1}	{3, 7}	{19, 11}	{13, 17}
{9, 1}	{9, 1}	{3, 7}	{19, 11}	{13, 17}
{3, 7}	{3, 7}	{9, 1}	{13, 17}	{19, 11}
{19, 11}	{19, 11}	{13, 17}	{9, 1}	{3, 7}
{13, 17}	{13, 17}	{19, 11}	{3, 7}	{9, 1}

(c) Find all subgroups of the group $U(20)/H$. (8 points)

Solution: The subgroups are $\{e\}$, $U(20)/H$, $\{\{3, 7\}, \{9, 1\}\}$, $\{\{19, 11\}, \{9, 1\}\}$, $\{\{13, 17\}, \{9, 1\}\}$.

3. Suppose that H and K are subgroups of a group G.
(a) Prove that $H \cap K$ is also a subgroup of G. (7 points)

Solution: Suppose that $a, b \in H \cap K$. Then $a, b \in H$ so $ab \in H$ because H is a group. Likewise, $a, b \in K$ so $ab \in K$. Therefore $ab \in H \cap K$ and so $H \cap K$ is closed under the operation. Certainly $e \in H$ and $e \in K$ so $e \in H \cap K$. Now if $a \in H \cap K$, then $a \in H$, so $a^{-1} \in H$. Likewise, $a^{-1} \in K$ so $a^{-1} \in H \cap K$ as desired.

(b) Further suppose that H and K are finite groups and also that $\gcd(|H|, |K|)$ is prime. Prove that $H \cap K$ is cyclic. (13 points)

Solution: Set $n = H \cap K$. Then since $H \cap K$ is a subgroup of both H and of K, n divides both $|H|$ and $|K|$. Thus n divides $\gcd(|H|, |K|)$, which is prime. Thus n is 1 or n is prime. If n is 1, then $H \cap K = \{e\}$ which is clearly cyclic. On the other hand, if $n = |H \cap K|$ is prime, then it is cyclic as we know.

4. Suppose that $\phi : A \to B$ is a homomorphism of groups. Show directly that there is a well defined injective group homomorphism $\overline{\phi} : A/\ker\phi \to B$ which sends the coset $a(\ker\phi)$ to $\phi(a)$. (20 points)

Solution: For simplicity, we write $K = \ker\phi$. First we show it is well defined so suppose $aK = a'K$. Then $\overline{\phi}(aK) = \phi(a)$ and $\overline{\phi}(a'K) = \phi(a')$ so we need to show $\phi(a) = \phi(a')$. Since $aK = a'K$, $a \in a'K$ so that $a = a'k$ for some $k \in K$. Therefore $\phi(a) = \phi(a'k) = \phi(a')\phi(k) = \phi(a')$ where the last equality follows from the fact that $k \in K = \ker\phi$. But this proves that $\overline{\phi}$ is well defined.

Now we show that $\overline{\phi}$ is a homomorphism. Now $\overline{\phi}((aK)(a'K)) = \overline{\phi}((aa')K) = \phi(aa') = \phi(a)\phi(a') = \overline{\phi}(aK)\overline{\phi}(a'K)$ as desired.

Finally we show $\overline{\phi}$ is injective. It is enough to show that $\ker\overline{\phi} = \{eK\}$. So suppose that $\overline{\phi}(aK) = e_B$, thus $\overline{\phi}(aK) = \phi(a) = e_B$ and so $a \in K$. Thus $aK = eK$ proving that $\overline{\phi}$ is injective as desired.

(EC) Suppose A and B are Abelian groups and $\varphi : A \to B$ is a group homomorphism. Suppose that there exists another group homomorphism $\psi : B \to A$ such that $\psi \circ \varphi = \mathrm{id}_A$. Prove that B is isomorphic to $A \oplus M$ for some other group M. (10 points)

Solution: We first identify A with its image in B (it is isomorphic to that image since φ is injective). Thus we have $A \subseteq B$ is a normal subgroup since B is Abelian. Set $M = B/A$ with natural surjective map $\pi : B \to B/A$. Consider the function

$$\eta : B \to A \oplus B/A$$

which sends b to $(\psi(b), \pi(b))$. Notice that

$$\eta(bb') = (\psi(bb'), \pi(bb')) = (\psi(b), \pi(b))(\psi(b'), \pi(b')) = \eta(b)\eta(b').$$

Therefore η is a homomorphism.

Now we show we show that η is injective. It is enough to show that $\ker \eta = e_B$. So suppose that $\eta(b) = (e_A, eA)$. Thus $\psi(b) = e_A$ and $bA = \pi(b) = eA$. The second equality implies that $b \in A$. But since $b \in A$, $\varphi(b) = b$ and so $\pi(b) = b$ as well. But then $b = e_A = e_B$ as desired.

Finally we show that η is surjective. Suppose that $(a, bA) \in A \oplus B/A$. Consider first $\psi(b)$. We know there exists an element $a' \in A$ such that $\psi(b)a' = a$ (set $a' = (\psi(b))^{-1}a$). Now consider $ba' = b\varphi(a') \in B$ where again we are identifying A with its image in B. Then

$$\eta(ba') = (\psi(ba'), (ba')A) = (\psi(b)a', (b(a'A))) = (a, bA)$$

show that η is surjective as desired.