WORKSHEET #5– MATH 3210
FALL 2019

DUE WEDNESDAY, OCTOBER 30TH

You may work in groups of up to 4 on this assignment. Only one assignment needs to be turned in per group. It still needs to be turned in on gradescope.

Recall a *partition P of* $[a,b]$ is a list of real numbers $x_0, \ldots, x_n$ with $a = x_0 < x_1 < \cdots < x_n = b$.

Suppose $f : [a,b] \longrightarrow \mathbb{R}$ is a bounded function and P is a partition of $[a,b]$. Set $M_k = \sup(f([x_{k-1}, x_k]))$ and $m_k = \inf(f([x_{k-1}, x_k]))$. We the define the *upper sum for f* to be:

$$U(f,P) = \sum_{k=1}^{n} M_k \cdot (x_k - x_{k-1})$$

Likewise we define the *lower sum* for f to be:

$$L(f,P) = \sum_{k=1}^{n} m_k \cdot (x_k - x_{k-1}).$$

1. Prove that with notation as above, for a fixed partition P, that

$$L(f,P) \leq U(f,P).$$

If for each $k = 1, \ldots, n$, we choose some (for instance random) $\overline{x_k} \in [x_{k-1}, x_k]$, then we can also form the sum

$$\sum_{k=1}^{n} f(\overline{x}_k) \cdot (x_k - x_{k-1})$$

2. How does the sum above compare to $U(f, P)$ and $L(f, P)$? Prove your answer.

3. Consider the function $f : [2, 5] \longrightarrow \mathbb{R}$ defined by $f(x) = x^2$.

(a) Create a partition P of $[2, 5]$ dividing it up into 3 different intervals.

(b) Compute the upper sum $U(f, P)$ and the lower sum $L(f, P)$.

(c) Compute $U(f, P) - L(f, P)$.

If $P = \{x_0, x_1, \ldots, x_k\}$ is a partition of $[a, b]$, and P' is another partition of $[a, b]$, then we say that P' is a *refinement of* P if $P \subseteq P'$. In other words, P' has all the intervals of P but some are subdivided.

4. Suppose that P' is a refinement of P. Prove that with notation as above,

$$L(f, P) \leq L(f, P').$$

Hint: Any refinement can be obtained by adding one element at a time, in succession. Prove it when P' adds a single element y_k with $x_{k-1} < y_k < x_k$ to P.

It is also true that $U(f, P') \leq U(f, P)$, but I'm not asking you to prove it.

5. Suppose that P, Q are two partitions of $[a, b]$. Prove that there is another partition R that is a refinement of both P and a refinement of Q.

Suppose $f : [a, b] \longrightarrow \mathbb{R}$ is bounded. Set:

$$\overline{\int_a^b} f(x)dx := \inf \big\{ U(f, Q) \mid Q \text{ is a partition of } [a, b] \big\} \text{ and } \underline{\int_a^b} f(x)dx := \sup \big\{ L(f, Q) \mid Q \text{ is a partition of } [a, b] \big\}$$

We say that f is *Riemann integrable* on $[a, b]$ if $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$. The integral $\displaystyle\int_a^b f(x)dx$ is the common value.

6. With notation as above, suppose that for every $\epsilon > 0$ there exists a partition P such that $U(f, P) - L(f, P) < \epsilon$. Prove that f is Riemann integrable.

Hint: If you can show that $\overline{\int}$ and $\underline{\int}$ are can be made arbitrarily close to each other (say within any ϵ), this proves they are equal. If you look in the text, make sure you are looking at the right part of the proof.

7. Use the criterion from **6.** to prove that $f(x) = 3x$ is integrable on $[1, 2]$.

Hint: Consider the partition P breaking $[1, 2]$ into n equal subintervals. Compute $U(f, P)$ and $L(f, P)$ for that P.