WORKSHEET #1 – MATH 3210, FALL 2019

DUE FRIDAY, AUGUST 30TH

You may work in groups of up to 4 on this assignment. Only one assignment needs to be turned in per group. It still needs to be turned in on gradescope.
Recall the following definition:

Definition: Let F be an ordered field (such as $\mathbb{R}$). Suppose that $S \subseteq F$ is a subset that is bounded above. That is there exists a $B \in F$ such that $B \geq s$ for all $s \in S$. In this case, we say that B is an upper bound for S. An element $b \in F$ is called a *least upper bound* (or *supremum*) for S is for every upper bound B of S, $b \leq B$.
Similarly, we define bounded below sets and the greatest lower bound (sometimes called the *infimum*).

Completeness Axiom (C): We say that an orderered field F is complete if every bounded above set has a least upper bound.

The real numbers: The real numbers are (any) complete ordered field.

1. Let $I = (-1, 1)$ and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be $f(x) = x^2$. Compute the least upper bound of $f(I)$.

2. With notation as in **1.**, compute the greatest lower bound of $f^{-1}(I)$.

Given any set $S \subseteq \mathbb{R}$, the *supremum* of S

$$\sup(S) = \begin{cases} \infty & \text{if } S \text{ is not bounded above} \\ \text{the least upper bound of } S & \text{if } S \text{ is bounded above.} \end{cases}$$

Likewise the *infimum* of S is

$$\inf(S) = \begin{cases} -\infty & \text{if } S \text{ is not bounded below} \\ \text{the greatest lower bound of } S & \text{if } S \text{ is bounded below.} \end{cases}$$

3. Compute the sup and inf of the following set $S = \{\frac{1}{n^2+1} \mid n \in \mathbb{Z}_{>0}\}$.

4. Suppose that $x \in \mathbb{R}$ and $S \subseteq \mathbb{R}$. Prove that

$$\sup S \leq x$$

if and only if $S \leq x$ for every $s \in S$.

5. Suppose that $x \in \mathbb{R}$ and $S \subseteq \mathbb{R}$. Prove that

$$\sup S > x$$

if and only if $s > x$ for some $s \in S$.

6. Suppose that $A \subseteq B \subseteq \mathbb{R}$. Prove that

(a) $\sup A \leq \sup B$ and also that

(b) $\inf B \leq \inf A$.

Hint: Make sure to handle appropriately the case when A, B are not bounded.

7. Suppose that A, B are two non-empty sets of real numbers. Prove that

$$\sup(A \cup B) = \max\{\sup A, \sup B\} \text{ and } \inf(A \cup B) = \min\{\inf A, \inf B\}.$$

Hint: Make sure to handle appropriately the case when A, B are not bounded.