WORKSHEET #8 – MATH 3210
FALL 2018

NOT DUE

You may work in groups of up to 4. This worksheet covers similar material to the upcoming midterm.

1. Short answer questions.

(a) Give a precise definition of the statement $\lim a_n = -\infty$.

(b) Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ is the function $f(x) = x^2 + 1$. Compute $f^{-1}(\{5\} \cup (1, 2])$.

(c) Suppose $S \subseteq \mathbb{R}$ is a bounded set. Give a precise definition of $\inf(S)$.

(d) Give an example of a non-continuous function $f : [0, 1] \longrightarrow \mathbb{R}$ so that $f([0, 1]) = [0, 1]$.

(e) Give a short proof that if $\lim a_n = K$ and $\lim b_n = L$, then $\lim(a_n + b_n) = K + L$.

2. More short answer questions.

(a) Give an example of an integrable function that is not continuous.

(b) Suppose $f : (a, b) \longrightarrow \mathbb{R}$ is differentiable. Use the Mean Value theorem to give a short proof of the fact that if $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.

(c) Give an example of a function $f : [1, 3] \longrightarrow \mathbb{R}$ and two partitions $P \subseteq Q$ of $[1, 3]$ so that $U(f, P) > U(f, Q)$.

(d) Suppose $\sum_{k=0}^{\infty} c_k(x-a)^k$ is a power series. Give a precise definition of this series' radius of convergence.

(e) Give an example of a convergent series that is not absolutely convergent.

(f) Suppose $I \subseteq \mathbb{R}$ is an interval. Precisely define what it means that for a sequence of functions $f_n : I \longrightarrow \mathbb{R}$ to uniformly converge to a function $f : I \longrightarrow \mathbb{R}$.

3. Even more short answer questions.

(a) Compute the radius of convergence of the power series $1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots = \sum_{k=0}^{\infty}(-1)^k(x-1)^k$.

(b) Derive a closed formula for the sum $\sum_{k=0}^{\infty} r^k$ for real numbers $|r| < 1$.

(c) Is the series $\sum_{k=2}^{\infty} \frac{(-1)^k k^2}{k!}$ absolutely convergent, conditionally convergent or divergent.

(d) Precisely state Taylor's theorem on infinite series.

(e) State the first fundamental theorem of calculus.

(f) Compute $\frac{d}{dx}\int_2^{\ln(x)} e^{t^2}\,dt$.

(g) Compute $\lim_{x \longrightarrow 0} \frac{(\sin(x))^2}{x\cos(1/x)}$ or show that it doesn't exist.

(h) Precisely state the Weierstrass M-test.

4. Even more short answer questions.

(a) Precisely state the archimedean property of the real numbers.

(b) Give an example of a differentiable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ whose derivative is not continuous.

(c) If $f : [a, b] \longrightarrow \mathbb{R}$ is integrable, is the function $g(x) = \int_a^x f(t)dt$ always continuous? Always differentiable?

(d) Is every analytic function differentiable?

(e) State the completeness axiom of the real numbers.

(f) Precisely define what it means for a sequence a_n to be Cauchy.

(g) Is every Cauchy sequence bounded?

(h) Precisely state the Bolzano Weierstrass theorem.

5. Suppose $\sum_{k=1}^{\infty} a_k$ is conditionally convergent (and in particular, not absolutely convergent). Let $a_{p(k)}$ is the subsequence of positive terms of the sequence a_k and $a_{n(k)}$ is the subsequence of non-positive terms of a_k. Show that $\sum_{k=1}^{\infty} a_{p(k)}$ diverges and $\sum_{k=1}^{\infty} a_{n(k)}$ diverges.

Hint: Suppose one converged to some finite number L. Derive a contradiction.

6. Suppose that $\sum_{k=1}^{\infty} a_k$ is a conditionally convergent series. Prove there is a reordering of the series that converges to ∞.

Hint: What you need to do is show there is a reordering so that for every K, there is an N, so that $\sum_{k=1}^{n} a_k > K$ for all $n > N$. One way to do this is to consider a sequence of integer Ks, maybe $= 1, 2, 3, 4, \ldots,$ and rig things so that the partial sums get bigger than each K (and stay bigger than each K).

7. Suppose that $f_n : [a, b] \longrightarrow \mathbb{R}$ is a sequence of continuous functions such that $f_n(x)$ converges to $f(x)$ uniformly. Show that

$$\lim_n \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

Hint: Choose $\epsilon > 0$. Choose N so that if $n > N$, then $|f_n(x) - f(x)| < \epsilon/(b-a)$ for $n > N$ for all $x \in [a, b]$. Consider $f_n(x) - \epsilon/(b-a) < f(x) < f_n(x) + \epsilon/(b-a)$, and integrate.

8. Use the previous exercise to show that if $f_k : [a, b] \longrightarrow \mathbb{R}$ is a sequence of continuous functions such that $\sum_k^\infty f_k(x)$ converges uniformly to a function $g(x)$, then $\sum_{k=1}^\infty \int_a^b f_k(t)dt = \int_a^b g(t)dt$.

9. Compute the limit (if you dare):

$$\lim_{x \longrightarrow 0} \left(\frac{\int_0^x te^{t^3}\, dt}{\sum_{k=0}^{\infty} k^2 x^k} \right)$$

10. Suppose that $\{a_n\}$ converges to 0. Define a new sequence b_n as follows:

$$b_n = \begin{cases} a_n & \text{if } n \text{ is odd} \\ 1/n & \text{if } n \text{ is even} \end{cases}$$

In other words $\{b_n\}$ is the sequence

$$\{a_1, 1/2, a_3, 1/4, a_5, 1/6, a_7, 1/8, \dots\}.$$

Prove that b_n also converges to 0.

11. Consider the function $f(x) = \frac{x+1}{x+3}$. Prove directly from the definition that $f(x)$ is continuous at $a = 2$.

12. Prove that if f is an infinitely differentiable function on $(a-r, a+r)$, and there is a constant K such that

$$|f^{(n)}(x)| \leq K\frac{n!}{r^n}$$

for all $n \in \mathbb{N}$ and all $x \in (a-r, a_r)$, then the Taylor series for f at a converges to f on $(a-r, a+r)$.

13. Find the Taylor series expansion for $\cos(x)$ at 0 and show it converges for all x.

14. Recall the following axioms for an ordered field F and arbitrary $x, y, z \in F$.

- A1 $x + y = y + x$.
- A2 $x + (y + z) = (x + y) + z$.
- A3 $\exists\, 0 \in F$ such that $0 + x = x$.
- A4 For each $x \in F$, $\exists -x \in F$ with $x + (-x) = 0$.
- M1 $xy = yx$.
- M2 $x(yz) = (xy)z$.
- M3 $\exists 1 \in F$ such that $1 \neq 0$ and $1x = x$.
- D $x(y + z) = xy + xz$.
- F If $x \neq 0$, then $\exists x^{-1} \in F$ so that $xx^{-1} = 1$.
- O1 Either $x \leq y$ or $y \leq x$.
- O2 If $x \leq y$ and $y \leq x$ then $x = y$.
- O3 If $x \leq y$ and $y \leq z$ then $x \leq z$.
- O4 If $x \leq y$ then $x + z \leq y + z$.
- O5 If $x \leq y$ and $0 \leq z$, then $xz \leq yz$.

Prove that if $x \leq y$ and then $-y \leq -x$ using only the axioms above. Please use complete sentences in your justification.

Hint: You aren't allowed to multiply by -1 and flip inequalities, use O4 instead.

15. Suppose that $(a, b) \subseteq \mathbb{R}$ is a non-empty open interval and that we have a differentiable function $f : (a, b) \longrightarrow \mathbb{R}$ such that $|f'(x)| < M$ for some constant M for all $x \in (a, b)$. Use the Mean Value Theorem to prove that f is uniformly continuous.

16. Suppose that a_n and b_n are sequences such that $\lim a_n = L$ and $\lim b_n = K$. Prove directly using the definition of the limit that

$$\lim(a_n \cdot b_n) = L \cdot K.$$