WORKSHEET #8 – MATH 3210
FALL 2018

NOT DUE

You may work in groups of up to 4. This worksheet covers similar material to the upcoming midterm.

1. Short answer questions.

(a) Give a precise definition of the statement $\lim a_n = -\infty$.

Solution: For every $K \in \mathbb{R}$, there exists an N so that if $n > N$, then $a_n < K$.

(b) Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ is the function $f(x) = x^2 + 1$. Compute $f^{-1}(\{5\} \cup (1, 2])$.

Solution:

$$\{-2, 2\} \cup (0, 1] \cup [-1, 0).$$

(c) Suppose $S \subseteq \mathbb{R}$ is a bounded set. Give a precise definition of $\inf(S)$.

Solution: $\inf(S)$ is the greatest lower bound of S.

(d) Give an example of a non-continuous function $f : [0, 1] \longrightarrow \mathbb{R}$ so that $f([0, 1]) = [0, 1]$.

Solution:

$$f(x) = \begin{cases} 2x & 0 \leq x \leq 1/2 \\ 0 & 1/2 < x \leq 1. \end{cases}$$

(e) Give a short proof that if $\lim a_n = K$ and $\lim b_n = L$, then $\lim(a_n + b_n) = K + L$.

Solution: By definition there exists N_1 so that if $n > N_1$, then $|a_n - K| < \epsilon/2$. Likewise there exists N_2 so that if $n > N_2$ then $|b_n - L| < \epsilon/2$. Suppose $N = \max(N_1, N_2)$ and $n > N$. Then we have that

$$|a_n + b_n - (K + L)| \leq |a_n - K| + |b_n - L| < \epsilon/2 + \epsilon/2 = \epsilon.$$

2. More short answer questions.

(a) Give an example of an integrable function that is not continuous.

Solution: $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$

(b) Suppose $f : (a, b) \longrightarrow \mathbb{R}$ is differentiable. Use the Mean Value theorem to give a short proof of the fact that if $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.

Solution: Choose $x < y$ with $x, y \in (a, b)$. Then by the mean value theorem, there exists $c \in (x, y)$ such that $f'(c) = \frac{f(y)-f(x)}{y-x}$. But $f'(c) = 0$ so that $f(y) - f(x) = 0$ and so $f(y) = f(x)$. Thus f is constant.

(c) Give an example of a function $f : [1, 3] \longrightarrow \mathbb{R}$ and two partitions $P \subseteq Q$ of $[1, 3]$ so that $U(f, P) > U(f, Q)$.

Solution: Let $P = \{1, 3\}$ and $Q = \{1, 2, 3\}$, and set $f(x) = x$. Then $U(f, P) = 3 \cdot (3 - 1) = 6$. But $U(f, Q) = 2 \cdot (2 - 1) + 3 \cdot (3 - 2) = 5$.

(d) Suppose $\sum_{k=0}^{\infty} c_k(x-a)^k$ is a power series. Give a precise definition of this series' radius of convergence.

Solution: The radius of convergence is the largest number R so that if $x \in (a-R, a+R)$, then the power series $\sum_{k=0}^{\infty} c_k(x - a)^k$ converges. It is also

$$R = \frac{1}{\limsup |c_k|^{1/k}}.$$

(e) Give an example of a convergent series that is not absolutely convergent.

Solution: Consider $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$.

(f) Suppose $I \subseteq \mathbb{R}$ is an interval. Precisely define what it means that for a sequence of functions $f_n : I \longrightarrow \mathbb{R}$ to uniformly converge to a function $f : I \longrightarrow \mathbb{R}$.

Solution: It means for every $\epsilon > 0$, there exists an N so that if $n > N$, then $|f_n(x) - f(x)| < \epsilon$ for every $x \in I$.

3. Even more short answer questions.

(a) Compute the radius of convergence of the power series $1-(x-1)+(x-1)^2-(x-1)^3+\cdots = \sum_{k=0}^{\infty}(-1)^k(x-1)^k$.

Solution: The radius is $R = \frac{1}{\limsup |(-1)^k|^{1/k}}$.

(b) Derive a closed formula for the sum $\sum_{k=0}^{\infty} r^k$ for real numbers $|r|<1$.

Solution: Consider the partial sums $s_n = \sum_{k=0}^{n} r^k$. Then $(1-r)s_n = s_n - rs_n = 1-r^{n+1}$. As n goes to infinity, this goes to 1. Hence the limit of partial sums equals $\frac{1}{1-r}$.

(c) Is the series $\sum_{k=2}^{\infty} \frac{(-1)^k k^2}{k!}$ absolutely convergent, conditionally convergent or divergent.

Solution: Absolutely convergent.

(d) Precisely state Taylor's theorem on infinite series.

Solution: Let f be a function that has continuous derivatives up through order $n+1$ at all points of an open interval $I=(a-D,a+D)$. Then for each $x\in I$,

$$f(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+R_n(x).$$

where $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some c between a and x.

(e) State the first fundamental theorem of calculus.

Solution: Suppose $a<b$ are real numbers. Suppose $f:[a,b]\longrightarrow\mathbb{R}$ is continuous and differentiable on (a,b) and that $f'(x)$ is integrable on $[a,b]$. Prove that

$$f(b)-f(a) = \int_a^b f'(x)dx.$$

(f) Compute $\frac{d}{dx}\int_2^{\ln(x)} e^{t^2}dt$.

Solution: By the fundamental theorem of calculus and the chain rule, the derivative is $e^{((\ln(x))^2)}\cdot(1/x)$.

(g) Compute $\lim_{x\longrightarrow 0}\frac{(\sin(x))^2}{x\cos(1/x)}$ or show that it doesn't exist.

Solution: Both the top and bottom go to zero as x does. Furthermore, the derivative of the top is $2\sin(x)\cos(x)$. The derivative of the bottom is $\cos(1/x)+x\sin(1/x)(-1/x^2)$. Now, as x goes to zero, the top still goes to zero. The bottom however doesn't converge to anything. In fact, it's even unbounded. In particular, L'Hôpital's does *NOT* apply. For x near 0 on the right, the numerator is bigger than $1/2x^2$. If we cancel the x we get $\frac{1/2x}{\cos(1/x)}$. As x gets close to zero, $\cos(1/x)$ gets arbitrarily close to zero, infinitely many times. So the function is unbounded.

Indeed, to make this precise, here's a sketch, fix some K. Notice that if $x=\frac{1}{n((\pi/2)+2\pi)-\epsilon^2}$, then $|\cos(1/x)| = |\cos(\pi/2-\epsilon^2)|<\epsilon^2$ (for $\epsilon>0$ sufficiently small, with $1/(2n\epsilon^2)>K$ for any K). Thus we, have

$$|\frac{(\sin(x))^2}{x\cos(1/x)}| \geq |\frac{1/2x^2}{x\cos(1/x)}| \geq |\frac{1/2x}{\epsilon^2}| > K.$$

which can be made arbitrarily large. (This was harder than I intended).

(h) Precisely state the Weierstrass M-test.

Solution: Consider a series of functions $\sum_{k=0}^{\infty} f_k(x)$, $f_k: I\longrightarrow\mathbb{R}$ for some interval I. Suppose we have M_k a sequence of real numbers such that $M_k\geq|f_k(x)|$ for all $x\in I$. Then if $\sum_{k=0}^{\infty}M_k$ converges, we have that $\sum_{k=0}^{\infty} f_k(x)$ converges uniformly on I.

4. Even more short answer questions.

(a) Precisely state the archimedean property of the real numbers.

Solution: For every real number R, there exist an integer $N > R$.

(b) Give an example of a differentiable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ whose derivative is not continuous.

Solution:

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

This works.

(c) If $f : [a, b] \longrightarrow \mathbb{R}$ is integrable, is the function $g(x) = \int_a^x f(t)dt$ always continuous? Always differentiable?

Solution: It is always continuous. It is not differentiable in general (one needs f to be continuous).

(d) Is every analytic function differentiable?

Solution: Yes.

(e) State the completeness axiom of the real numbers.

Solution: Every bounded above set has a least supper bound (a supremum).

(f) Precisely define what it means for a sequence a_n to be Cauchy.

Solution: It means that for every $\epsilon > 0$, there exists an N so that if $n, m > N$, we have that $|a_n - a_m| < \epsilon$.

(g) Is every Cauchy sequence bounded?

Solution: Yes (every Cauchy sequences converges and hence is bounded).

(h) Precisely state the Bolzano Weierstrass theorem.

Solution: Suppose that $\{a_n\}$ is a bounded sequence. Then $\{a_n\}$ has a convergent subsequence.

5. Suppose $\sum_{k=1}^{\infty} a_k$ is conditionally convergent (and in particular, not absolutely convergent). Let $a_{p(k)}$ is the subsequence of positive terms of the sequence a_k and $a_{n(k)}$ is the subsequence of non-positive terms of a_k. Show that $\sum_{k=1}^{\infty} a_{p(k)}$ diverges and $\sum_{k=1}^{\infty} a_{n(k)}$ diverges.

Hint: Suppose one converged to some finite number L. Derive a contradiction.

Solution: If both the negative terms converge, and the positive terms converge, then the series is absolutely convergent. So we need to rule out the case where one converges and one diverges.

Suppose first that $\sum_{k=1}^{\infty} a_{p(k)}$ converges to $L \in \mathbb{R}$. Then it is absolutely convergent (since it is made up of positive numbers). Suppose for a contradiction that sum of negative integers diverges. Thus there exists an M_1 so that if $m > M_1$, we have that $5 > L - \sum_{i=1}^{m} a_{p(i)} = \sum_{i=m+1}^{\infty} a_{p(i)}$. Choose an integer K so that $K > p(k)$ for all $k \leq N$. Since the sum of the $a_{n(k)}$ diverge, we see that $\sum_{k,n(k)>K} a_{n_k} = -\infty$, and in particular, for each real number R, we can find some D so that $\sum_{k,n(k)>K}^{n(k)<D} a_{n(k)} < R - 5$. Then $\sum_{k>K}^{k<D} a_k$ is the same finite sum, but it also has finitely many positive terms, whose sum is less than 5. Thus $\sum_{k>K}^{k<D} a_k < R - 5 + 5 = R$. In particular, the partial sums of are unbounded. Thus the series $\sum_{k=0}^{\infty} a_k$ diverges.

By symmetry, we also have a contradiction if the sum of the negative terms converges,

6. Suppose that $\sum_{k=1}^{\infty} a_k$ is a conditionally convergent series. Prove there is a reordering of the series that converges to ∞.

Hint: What you need to do is show there is a reordering so that for every K, there is an N, so that $\sum_{k=1}^{n} a_k > K$ for all $n > N$. One way to do this is to consider a sequence of integer Ks, maybe $= 1, 2, 3, 4, \ldots$, and rig things so that the partial sums get bigger than each K (and stay bigger than each K).

Solution: Let $a_{n(k)}$ be the sequence of negative terms and $a_{p(k)}$ be the sequence of positive terms. By the previous exercise, we know the sum of the positive terms is infinite, as is the sum of the negative terms. Consider a sequence of integers $K = 1, 2, 3, \ldots$ as in the hint. First we choose the smallest s_1 so that $a_{p(1)}, \ldots, a_{p(s_1)}$ so that $a_{p(1)} + \cdots + a_{p(s_1)} + a_{n(1)} > 1$. Next choose the smallest $s_2 > s_1$ so that $a_{p(1)} + \cdots + a_{p(s_1} + a_{n(1)} + a_{p(s_1+1)} + \ldots a_{p(s_2)} + a_{n(2)} > 2$. Continue in this way.

7. Suppose that $f_n : [a,b] \longrightarrow \mathbb{R}$ is a sequence of continuous functions such that $f_n(x)$ converges to $f(x)$ uniformly. Show that

$$\lim_n \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

Hint: Choose $\epsilon > 0$. Choose N so that if $n > N$, then $|f_n(x) - f(x)| < \epsilon/(b-a)$ for $n > N$ for all $x \in [a,b]$. Consider $f_n(x) - \epsilon/(b-a) < f(x) < f_n(x) + \epsilon/(b-a)$, and integrate.

Solution: Following the hint, we integrate that set of inequalities to obtain for $n > N$.

$$\int_a^b f_n(x)dx - \epsilon = \int_a^b (f_n(x) - \epsilon/(b-a))dx \leq \int_a^b f(x)dx \leq \int_a^b (f_n(x) + \epsilon/(b-a))dx = \int_a^b f_n(x)dx + \epsilon$$

Hence we have that $\left| \int_a^b f_n(x)dx - \int_a^b f(x)dx \right| < \epsilon$ for $n > N$, as desired.

8. Use the previous exercise to show that if $f_k : [a,b] \longrightarrow \mathbb{R}$ is a sequence of continuous functions such that $\sum_k^\infty f_k(x)$ converges uniformly to a function $g(x)$, then $\sum_{k=1}^\infty \int_a^b f_k(t)dt = \int_a^b g(t)dt$.

Solution: We know that the sequence of partial sums $s_n(x) = \sum_{k=0}^n f_k(x)$ is a sequence of functions which converges uniformly. By the previous exercise, we have that

$$\lim \int_a^b \left(\sum_{k=0}^n f_k(x) \right) dx = \lim \int_a^b s_n(x)dx = \int_a^b \left(\sum_{k=0}^\infty f_k(x) \right) dx.$$

But

$$\lim \int_a^b \left(\sum_{k=0}^n f_k(x) \right) dx = \lim \sum_{k=0}^n \int_a^b f_k(x)dx$$

since the sum is finite. This completes the proof.

9. Compute the limit (if you dare):

$$\lim_{x\longrightarrow 0}\left(\frac{\int_0^x te^{t^3}\,dt}{\sum_{k=0}^{\infty} k^2x^k}\right)$$

Solution: First note that the bottom is differentiable on $(-1,1)$. We use L'Hôpital's rule. The top becomes xe^{x^3}. The bottom becomes $\sum_{k=0}^{\infty}(k+1)^3x^k$. Now the top goes to zero, but the bottom goes to 1, so the limit is 0.

10. Suppose that $\{a_n\}$ converges to 0. Define a new sequence b_n as follows:

$$b_n = \begin{cases} a_n & \text{if } n \text{ is odd} \\ 1/n & \text{if } n \text{ is even} \end{cases}$$

In other words $\{b_n\}$ is the sequence

$$\{a_1, 1/2, a_3, 1/4, a_5, 1/6, a_7, 1/8, \ldots\}.$$

Prove that b_n also converges to 0.

Solution: Fix $\epsilon > 0$. Since a_n converges to 0, there exists a N_1 so that if $n > N_1$, then $|a_n| < \epsilon$. On the other hand, if we set $N_2 = \frac{1}{\epsilon}$, then if $n > N_2$, we see that $|1/n| < 1/N_2 = \epsilon$. Therefore, set $N = \max(N_1, N_2)$ and choose $n > N$. If n is odd, then $|b_n| = |a_n| < \epsilon$ since $n > N_1$. If n is even, then $|b_n| = 1/n < \epsilon$ since $n > N_2$.

11. Consider the function $f(x) = \frac{x+1}{x+3}$. Prove directly from the definition that $f(x)$ is continuous at $a = 2$.

Solution: We begin with some scratch work. Note $f(2) = \frac{3}{5}$. We want

$$\epsilon > \left|\frac{x+1}{x+3} - \frac{3}{5}\right| = \left|\frac{5x+5-3x-9}{3x+9}\right| = \frac{2}{5} \cdot \left|\frac{x-2}{x+3}\right|.$$

First suppose $\delta \leq 1$. Then $x+3$ is smaller than $(2+1)+3 = 6$ and bigger than $(2-1)+3 = 4$ if $|x-2| < \delta$ (only the bigger is relevant). In this case,

$$\frac{2}{5} \cdot \left|\frac{x-2}{x+3}\right| \leq \frac{2}{5} \cdot \frac{\delta}{4} = \delta/10.$$

This finishes the scratch work and we can set $\delta = 10 \cdot \epsilon$.

Now we do the real proof. Choose $\epsilon > 0$, set $\delta = \min(1, 10 \cdot \epsilon)$ and suppose that $|x-2| < \delta \leq 1$. In this case notice that $x+3 > 4$. Then

$$|f(x) - f(2)| = \left|\frac{x+1}{x+3} - \frac{3}{5}\right| = \frac{2}{5} \cdot \left|\frac{x-2}{x+3}\right| \leq \frac{2}{5} \cdot \frac{\delta}{4} = \delta/10 = \epsilon,$$

as desired.

12. Prove that if f is an infinitely differentiable function on $(a-r, a+r)$, and there is a constant K such that

$$|f^{(n)}(x)| \leq K\frac{n!}{r^n}$$

for all $n \in \mathbb{N}$ and all $x \in (a-r, a_r)$, then the Taylor series for f at a converges to f on $(a-r, a+r)$.

Solution: By Taylor's formula, it suffices to show that $R_n(x)$ goes to zero for all $x \in (a-r, a+r)$. So fix such an x and recall that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x. Then by our hypothesis,

$$R_n(x) \leq K\frac{(n+1)!/r^{n+1}}{(n+1)!}(x-a)^{n+1} = K\left(\frac{x-a}{r}\right)^{n+1}.$$

But $x - a < r$ and so this goes to zero, as claimed.

13. Find the Taylor series expansion for $\cos(x)$ at 0 and show it converges for all x.

Solution: It is easy to compute that the Taylor series is

$$\frac{1 - x^2/2 + x^4/4! - x^6/6! + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k+2}{2}(-1)^{\lfloor k/2 \rfloor} x^k}{k!}$$

Note the term $\frac{(-1)^k+2}{2}$ alternates between being 0 and 1 while $(-1)^{\lfloor k/2 \rfloor}$ is $1, 1, -1, -1, 1, 1, -1, -1, \ldots$. Regardless, $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$, where $f(x) = \cos(x)$. Since derivatives of $\cos(x)$ are plus or minus cos or sin of x, we see that $|R_n(x)| \leq \frac{x^{n+1}}{(n+1)!}$, which goes to zero for any fixed x, as n goes to infinity. This shows that our Taylor series above converges.

14. Recall the following axioms for an ordered field F and arbitrary $x, y, z \in F$.

- A1 $x + y = y + x$.
- A2 $x + (y + z) = (x + y) + z$.
- A3 $\exists\, 0 \in F$ such that $0 + x = x$.
- A4 For each $x \in F$, $\exists -x \in F$ with $x + (-x) = 0$.
- M1 $xy = yx$.
- M2 $x(yz) = (xy)z$.
- M3 $\exists 1 \in F$ such that $1 \neq 0$ and $1x = x$.
- D $x(y + z) = xy + xz$.
- F If $x \neq 0$, then $\exists x^{-1} \in F$ so that $xx^{-1} = 1$.
- O1 Either $x \leq y$ or $y \leq x$.
- O2 If $x \leq y$ and $y \leq x$ then $x = y$.
- O3 If $x \leq y$ and $y \leq z$ then $x \leq z$.
- O4 If $x \leq y$ then $x + z \leq y + z$.
- O5 If $x \leq y$ and $0 \leq z$, then $xz \leq yz$.

Prove that if $x \leq y$ and then $-y \leq -x$ using only the axioms above. Please use complete sentences in your justification.

Hint: You aren't allowed to multiply by -1 and flip inequalities, use O4 instead.

Solution: Adding $-x$ (which exists by A4) to both sides of $x \leq y$ and using O4, we see that $x - x \leq y - x$. Now adding $-y$ to both sides we see that $-y + x - x \leq -y + y - x$. Using $A4$ again (as well (A2) implicitly), we see that $-y + x - x = -y + 0 = -y$ and $-y + y - x = 0 - x = -x$. Thus $-y \leq -x$ as claimed.

15. Suppose that $(a, b) \subseteq \mathbb{R}$ is a non-empty open interval and that we have a differentiable function $f : (a, b) \longrightarrow \mathbb{R}$ such that $|f'(x)| < M$ for some constant M for all $x \in (a, b)$. Use the Mean Value Theorem to prove that f is uniformly continuous.

Solution: Choose $\epsilon > 0$ and set $\delta = \epsilon/M$ suppose that $x, y \in (a, b)$ with $|x - y| < \delta$. Without loss of generality, we may assume that $x < y$. By the Mean Value Theorem, there exists $c \in (x, y)$ so that $f'(c) = \frac{f(y)-f(x)}{y-x}$ and so $M(y - x) > |f(y) - f(x)|$ using that $|f'(c)| < M$. Hence

$$|f(y) - f(x)| < M(y - x) < M\delta = M \cdot \epsilon/M = \epsilon$$

as desired.

16. Suppose that a_n and b_n are sequences such that $\lim a_n = L$ and $\lim b_n = K$. Prove directly using the definition of the limit that

$$\lim(a_n \cdot b_n) = L \cdot K.$$

Solution: Again we start with scratch work.

$$|a_n \cdot b_n - L \cdot K| = |a_n \cdot b_n - a_n \cdot K + a_n \cdot K - L \cdot K| \leq |a_n| \cdot |b_n - K| + |a_n - L| \cdot |K|.$$

Now we come to the proof. Choose $\epsilon > 0$ Since a_n converges, it is bounded, so $|a_n| < M$ for all n. We can also choose N_1 so that if $n > N_1$, then $|a_n - L| < \epsilon/(2|K|)$. Finally choose N_2 so that if $n > N_2$, we have that $|b_n - K| < \epsilon/(2M)$. Set $N = \max(N_1, N_2)$ and suppose that $n > N$. Then

$$|a_n \cdot b_n - L \cdot K| = |a_n \cdot b_n - a_n \cdot K + a_n \cdot K - L \cdot K| \leq |a_n| \cdot |b_n - K| + |a_n - L| \cdot |K| < M\epsilon/(2M) + K\epsilon/(2K) = \epsilon$$

as desired.