WORKSHEET #6 – MATH 3210
FALL 2018

SOLUTIONS

You may work in groups of up to 4. This worksheet covers similar material to the upcoming midterm.

1. Short answer questions.

(a) Give an example differentiable function whose derivative is not continuous.

Solution: $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$.

(b) Give a precise definition of the expression $\lim_{x \longrightarrow \infty} f(x) = L$.

Solution: It means that for every $\epsilon > 0$ there exists a $K \in \mathbb{R}$ so that if $x > K$ (and is in the domain of f), we have that $|f(x) - L| < \epsilon$.

(c) Suppose $f : (a, b) \longrightarrow \mathbb{R}$ is differentiable at $c \in (a, b)$, give a short proof that f is continuous at c.

Solution: We need to show that $\lim_{x \longrightarrow c} f(x) = f(c)$ or in other words that $\lim_{x \longrightarrow c}(f(x) - f(c)) = 0$. But

$$\lim_{x \longrightarrow c} \left((f(x) - f(c)) \cdot \frac{(x-c)}{(x-c)} \right) = \lim_{x \longrightarrow c} \left(\frac{f(x) - f(c)}{x - c} \cdot (x - c) \right) = f'(c) \cdot 0 = 0.$$

(d) Suppose that $f(x) = e^{\sin(x)}$ with domain $(-\pi/2, \pi/2)$. Compute $(f^{-1})'(1)$.

Solution: First note that $f^{-1}(1) = 0$. So $(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)}$. But $f'(x) = e^{\sin(x)} \cos(x)$ and so $f'(0) = 1$ and so $(f^{-1})'(1) = 1/1 = 1$.

(e) Suppose $f : [a, b] \longrightarrow \mathbb{R}$ is continuous. Precisely define what it means for $c \in [a, b]$ to be a critical point.

Solution: Recall that c is a critical point if $f'(c) = 0$, if f is not differentiable at c, or c equals a or b.

2. More short answer questions.

(a) Precisely state the Mean Value Theorem.

Solution: Suppose $f : [a, b] \longrightarrow \mathbb{R}$ is continuous and differentiable on (a, b). Then there exists $c \in (a, b)$ so that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

(b) Suppose $f : (a, b) \longrightarrow \mathbb{R}$ is differentiable. Use the Mean Value theorem to give a short proof of the fact that if $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing.

Solution: We need to show that for $x < y \in (a, b)$, we have that $f(x) < f(y)$. Now apply the mean value theorem to $[x, y]$. Then there exists $c \in (x, y)$ such that

$$0 < f'(c) = \frac{f(y) - f(x)}{y - x}.$$

Since $y - x$ is positive, we have $f(y) - f(x) > 0$ and so $f(y) > f(x)$ as claimed.

(c) Precisely state Cauchy's Mean Value theorem.

Solution: Suppose that $f, g : [a, b] \longrightarrow \mathbb{R}$ are continuous functions which are differentiable on (a, b) and such that $g'(x) \neq 0$ for all $x \in (a, b)$. Then there exists $c \in (a, b)$ so that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

(d) Use L'Hôpital's rule to compute $\lim_{x \longrightarrow 1} \frac{\ln(x)}{x^1 - 1}$.

Solution: First notice that as x goes to 1, both the numerator and denominator go to zero. Thus we can apply L'Hôpital's rule and we need to compute $\lim_{x \longrightarrow 1} \frac{1/x}{1}$ which goes to 1, as desired.

(e) Precisely state what it means for a function $f : [a, b] \longrightarrow \mathbb{R}$ to be Riemann integrable on $[a, b]$.

Solution: It means that

$$\inf\{U(f, P) \mid P \text{ a partition of } [a, b]\} = \sup\{L(f, P) \mid P \text{ a partition of } [a, b]\}.$$

There are other correct statements as well. For instance, for every ϵ, there exists a partition P so that $U(f, P) - L(f, P) < \epsilon$.

(f) Suppose that $f : [a, b] \longrightarrow \mathbb{R}$ is a function and that $P \subseteq Q$ are partitions. Order the following from smallest to largest using $\leq$. $U(f, P)$, $U(f, Q)$, $L(f, P)$, $L(f, Q)$.

Solution: We have $L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$.

3. Even more short answer questions.

(a) Give an example of a non-integrable function $f : [0,1] \longrightarrow \mathbb{R}$.

Solution: The function $f(x) = \begin{cases} 0 & x \text{ is rational} \\ 1 & x \text{ is irrational} \end{cases}$ works.

(b) If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable. Is it always true that f is integrable on every closed bounded interval $[a, b]$.

Solution: Yes. Differentiable implies continuous. Continuous implies integrable.

(c) Give an example of a function $f : [a, b] \longrightarrow \mathbb{R}$ such that f is not integrable but $|f|$ is integrable.

Solution: The function $f(x) = \begin{cases} -1 & x \text{ is rational} \\ 1 & x \text{ is irrational} \end{cases}$ works since $|f(x)|$ is the constant function 1.

(d) Give a definition of the natural log using an integral.

Solution: $\ln(x) = \int_1^x \frac{1}{t} dt$.

(e) Precisely state the second fundamental theorem of calculus.

Solution: Suppose f is integrable on $[b, c]$. For any $a, x \in [b, c]$ define

$$F(x) = \int_a^x f(t)dt.$$

Then F is continuous on $[b, c]$ and furthermore, if f is continuous at $x \in (b, c)$, then F is differentiable at x and $F'(x) = f(x)$.

(f) Derive the formula for integration by parts using the product rule.

Solution: We know that $(fg)' = f'g + fg'$. Then write $(fg)' - fg' = f'g$. Taking the integral of both sides to get

$$\int_a^b (f(x)g(x))'dx - \int_a^b f(x)g'(x)dx = \int_a^b f'(x)g(x)dx.$$

But this is just

$$f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)dx = \int_a^b f'(x)g(x)dx.$$

(g) Precisely define what it means for an infinite series $\sum_{k=0}^{\infty} a_k$ to be convergent.

Solution: It means that if $s_n = \sum_{k=0}^{n} a_k$, then $\lim_{n \longrightarrow \infty} s_n = L$ for some $L \in \mathbb{R}$.

(h) Is the following infinite series convergent? $\sum_{k=1}^{\infty} \frac{2^{k+2}}{3^k}$.

Solution: Yes, it can be manipulated into a convergent geometric series.

4. Suppose that $f : [a, b] \longrightarrow \mathbb{R}$ is continuous. Show that there is a point $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx.$$

Hint: Use the Mean Value Theorem applied to the function $F(x) = \int_a^x f(t)dt$.

Solution: We know that F is differentiable on $[a, b]$ by the fundamental theorem of calculus. Thus there exists $c \in (a, b)$ so that

$$f(c) = F'(c) = \frac{F(b) - F(a)}{b-a}.$$

But $F(a) = 0$ and $F(b) = \int_a^b f(t)dt$ and so

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

as claimed.

5. Suppose that $f : [a, b] \longrightarrow \mathbb{R}$ is monotone non-increasing. Prove that f is integrable without citing Theorem 5.2.1 in the text.

Solution: We're not going to cite that theorem, we'll just reprove it. Choose a partition $P_n = \{x_0, \ldots, x_n\}$ of $[a, b]$ with equally spaced intervals. Then $U(f, P_n) = \sum_{k=1}^n f(x_{k-1})(x_k - x_{k-1})$ and $L(f, P_n) == \sum_{k=1}^n f(x_k)(x_k - x_{k-1})$. Hence, since $x_i - x_{i-1} = \frac{b-a}{n}$, $U(f, P_n) - L(f, P_n) = (f(x_0) - f(x_n)) \cdot \frac{b-a}{n}$. However, as $n \longrightarrow \infty$, this number goes to zero. Thus there is a sequence of partitions P_n with $U(f, P_n) - L(f, P_n)$ limiting to zero, and so f is integrable.

6. Prove directly from the definition that the Riemann integral of $f(x) = 2x + 1$ exists on the interval $[0, 1]$.

Solution: Let $P_n = \{0, 1/n, 2/n, \ldots, n/n = 1\}$ be a partition of $[0, 1]$. Then

$$U(f, P) = \sum_{k=1}^{n} f(k/n) \cdot (1/n) \text{ and } L(f, P) = \sum_{k=1}^{n} f((k-1)/n) \cdot (1/n)$$

Thus $U(f, P) - L(f, P) = (f(1) - f(0)) \cdot (1/n) = (3 - 1) \cdot (1/n) = \frac{2}{n}$. Since this goes to zero as n goes to infinity, the function is integrable.

7. Prove carefully that if $f : (a, b) \longrightarrow \mathbb{R}$ is a differentiable function with bounded derivative, then f is uniformly continuous.

Solution: Choose $\epsilon > 0$ and suppose that $|f'(c)| < M$ for $c \in (a, b)$. By the mean value theorem, for any $x < y$ in (a, b), there exists a $c \in (x, y)$ so that

$$f'(c) = \frac{f(y) - f(x)}{y - x}.$$

Therefore $M \cdot |y - x| > |f(y) - f(x)|$. Now, set $\delta = \epsilon/M$ and suppose that $|y - x| < \delta$, then

$$\epsilon = M \cdot \delta > M \cdot |y - x| > |f(y) - f(x)|$$

and f is uniformly continuous.

8. Prove that the function $f(x) = \begin{cases} x^2 & x \geq 0 \\ x\sin(x) & x \leq 0 \end{cases}$ is differentiable everywhere. You may use that $\sin(x)$ is differentiable with derivative $\cos(x)$.

Solution: For $x \neq 0$, it is differentiable since it is either polynomial or a product of polynomial functions. Thus it suffices to show that $\lim_{x \longrightarrow 0} \frac{f(x)-f(0)}{x}$ exists. However, we can prove that the left and right limits exist an agree. Notice that

$$\lim_{x \longrightarrow 0^+} \frac{f(x) - f(0)}{x} = \lim_{x \longrightarrow 0^+} \frac{x^2}{x} = \lim_{x \longrightarrow 0^+} x = 0$$

and

$$\lim_{x \longrightarrow 0^-} \frac{f(x) - f(0)}{x} = \lim_{x \longrightarrow 0^-} \frac{x\sin(x)}{x} = \lim_{x \longrightarrow 0^+} \sin(x) = 0.$$

9. Use the definition of the limit to prove that

$$3 = \lim_{x \longrightarrow 2} x^2 - 1.$$

Solution: Choose $\epsilon > 0$. We need to find $\delta > 0$ so that if $|x - 2| < \delta$ (and $x \neq 2$) then $|x^2 - 1 - 3| < \epsilon$. Let $\delta = \min(1, \epsilon/5)$ and suppose that $x \neq 2$ satisfies $|x - 2| < \delta$. Then $|x - 2| < 1$ and so $1 < x < 3$. Therefore $|x + 2| < 5$ and so

$$|x^2 - 1 - 3| = |x^2 - 4| = |x - 2| \cdot |x + 2| < |x - 2| \cdot 5 < \delta \cdot 5 \leq \epsilon/5 \cdot 5 = \epsilon$$

as desired.

10. Compute the derivative (with respect to x) of the function

$$g(x) = \int_{\sin(x)+2}^{e^x} \frac{1}{t} dt.$$

Hint: Break up the integral into two different integrals.

Solution: We can write

$$g(x) = \left(\int_1^{e^x} \frac{1}{t} dt \right) + \left(\int_{\sin(x)+2}^{1} \frac{1}{t} dt \right) = \left(\int_1^{e^x} \frac{1}{t} dt \right) - \left(\int_1^{\sin(x)+2} \frac{1}{t} dt \right)$$

By the chain rule and funadmental theorem of calculus, the derivative of $\int_1^{e^x} \frac{1}{t} dt$ is simply $\frac{1}{e^x} \cdot e^x = 1$. Likewise the derivative of $\int_1^{\sin(x)+2} \frac{1}{t} dt$ is $\frac{1}{\sin(x)+2} \cdot \cos(x)$. Hence

$$g'(x) = 1 - \frac{\cos(x)}{\sin(x)+2}.$$

11. Suppose that $f : [a, b] \longrightarrow \mathbb{R}$ any function and $f(x) \geq m$ for all $x \in [a, b]$. Prove directly that

$$\underline{\int_a^b} f(x) d(x) \geq m(b-a).$$

Solution: Choose a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$. Then $L(f, P) = \sum_{i=1}^n m_k(x_k - x_{k-1})$ using the notation from the book. But $m_k \geq m$ since m is a lower bound for m everywhere on $[a, b]$ and so

$$L(f, P) \geq \sum_{i=1}^n m(x_k - x_{k-1}) = m(b-a).$$

However, $\underline{\int_a^b} f(x) d(x) \geq L(f, P) \geq m(b-a)$ and the proof is complete.