WORKSHEET #4 – MATH 3210
FALL 2018

NOT DUE

This worksheet is roughly similar in format to the midterm (although it is much longer than the midterm).

1. Short answer.

(a) Give an example of a bounded sequence that is not convergent.

(b) Precisely define what it means for a sequence of functions $\{f_n : D \longrightarrow \mathbb{R}\}$ to be uniformly convergent.

(c) Precisely state the Bolzano Weierstrass theorem.

(d) If $f : [a, b] \longrightarrow \mathbb{R}$ is a continuous function, is it always the case that $\inf f([a, b]) \in f([a, b])$?

(e) Precisely define what it means for a sequence to be Cauchy.

(f) Precisely define what it means for a function to be uniformly continuous.

1. continued.

(g) If $a_n \longrightarrow a$ is a convergent sequence in $\mathbb{R}$, is $\{a_n\}$ always bounded?

(h) If $A \subseteq B$ are sets of real numbers, is it always true that $\sup A \geq \sup B$ or always true that $\sup B \geq \sup A$?

(i) Give a precise statement of the completeness axiom for the real numbers.

(j) Is it true that the rationals $\mathbb{Q}$ are an *ordered field*?

(k) Give an example of a domain D and a function $f : D \longrightarrow \mathbb{R}$ such that f is not uniformly continuous.

(l) Give an example of a convergent sequence that is not monotone.

(m) Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ is the function $f(x) = 1 + 2x$. Let $A = \{x \in \mathbb{R} \mid x^2 \leq 2\}$. Compute $f(A)$.

1. continued.

(n) Give an example of a convergent subsequence of a non-convergent sequence.

(o) Suppose $\{a_n\}$ is a sequence of real numbers. Precisely define what it means that $\lim a_n = -\infty$.

(p) If $a_n \longrightarrow a$ and $b_n \longrightarrow b$ are convergent sequences of real numbers, is it always true that $(a_n + b_n) \longrightarrow (a + b)$?

(q) Briefly prove that if $f : D \longrightarrow \mathbb{R}$ and $g : D \longrightarrow \mathbb{R}$ are continuous that $f + g$ is also continuous using the fact that if a function $h : D \longrightarrow \mathbb{R}$ is continuous if and only if for every convergent sequence $a_n \longrightarrow a$ of elements in D with $a \in D$, we have that $h(a_n) \longrightarrow h(a)$.

(r) Is every continuous function on the half-open interval $[0, 1)$ necessarily uniformly continuous?

(s) Consider the sets $B = \mathbb{Z}, C = [0, \pi]$. Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ is the function $f(x) = x^2$. Compute $f^{-1}(B \cap C)$.

(t) Suppose $\{a_n\}$ is a bounded sequence. Precisely define $\liminf\{a_n\}$.

2. Use the definition to prove that the sequence $\frac{n}{3n+(-1)^n}$ converges to $1/3$.

3. Use the definition to prove that the function $f(x) = x^2 + 1$ is continuous at $x = -2$.

4. Suppose that the sequence $\{a_n\}$ converges to a number L. Prove directly from the definition that $\{a_n\}$ is Cauchy.

5. Suppose that $\{a_n\}$ is a bounded sequence and $\{b_n\}$ converges to a number L. Assume further that both $a_n, b_n \geq 0$ for all n. Prove that

$$\limsup(a_n b_n) = L \cdot \limsup(a_n).$$

6. Give a direct proof using the definition that the function $f(x) = \sqrt{x}$ is continuous at all $a \geq 0$.

7. Suppose that $f : [0,1] \longrightarrow \mathbb{R}$ is continuous and $f(x) \leq 7$ for all $x \in [0,1)$. Prove that $f(1) \leq 7$.
Hint: One way to do it is to use the intermediate value theorem.

8. Suppose that $f : (a, c) \longrightarrow \mathbb{R}$ is continuous. Further suppose that $a < b < c$ and f is uniformly continuous on $(a, b]$ and also uniformly continuous on $[b, c)$. Prove that f is uniformly continuous on all of (a, c).

9. Suppose that $\{f_n : D \longrightarrow \mathbb{R}\}$ is a sequence of uniformly continuous functions that converge uniformly to a function $f : D \longrightarrow \mathbb{R}$. Prove that f is uniformly continuous.

Recall the following axioms for an ordered field F and arbitrary $x, y, z \in F$.

- A1 $x + y = y + x$.
- A2 $x + (y + z) = (x + y) + z$.
- A3 $\exists\, 0 \in F$ such that $0 + x = x$.
- A4 For each $x \in F$, $\exists\, -x \in F$ with $x + (-x) = 0$.
- M1 $xy = yx$.
- M2 $x(yz) = (xy)z$.
- M3 $\exists 1 \in F$ such that $1 \neq 0$ and $1x = x$.
- D $x(y + z) = xy + xz$.
- F If $x \neq 0$, then $\exists x^{-1} \in F$ so that $xx^{-1} = 1$.
- O1 Either $x \leq y$ or $y \leq x$.
- O2 If $x \leq y$ and $y \leq x$ then $x = y$.
- O3 If $x \leq y$ and $y \leq z$ then $x \leq z$.
- O4 If $x \leq y$ then $x + z \leq y + z$.
- O5 If $x \leq y$ and $0 \leq z$, then $xz \leq yz$.

10. Prove that if $x \leq y$ then $-y \leq -x$ using only the axioms above.

Hint: You aren't allowed to multiply by -1 and flip inequalities, use O4 instead.

11. Suppose that A, B are two non-empty and bounded sets of real numbers. Prove directly from the definition that

$$\inf(A \cup B) = \min\{\inf A, \inf B\}.$$