WORKSHEET #2 – MATH 3210
FALL 2018

DUE MONDAY SEPTEMBER 17TH

You may work in groups of up to 4. Only one worksheet is required per group.

We recall some definitions.

Definition. A sequence $\{a_n\}$ is called *bounded* if there exists an $K \geq 0$ such that $|a_n| < K$ for all n. It is *bounded above* if there is a $K \in \mathbb{R}$ such that $a_n < K$ for all n. It is *bounded below* if there is a $K \in \mathbb{R}$ such that $a_n > K$ for all n.

Definition. A sequence $\{a_n\}$ is called *non-decreasing* if $a_n \leq a_{n+1}$ for all n. A sequence is called *non-increasing* if $a_n \geq a_{n+1}$ for all n. If a sequence is either non-increasing or non-decreasing, we call the sequence *monotone.*

1. Write down examples of the following (proofs are not required):

(a) A non-decreasing sequence that is not bounded. (2 points)

Solution: $a_n = n$ works.

(b) An bounded sequence that is both not non-decreasing and not non-increasing. (2 points)

Solution: $a_n = (-1/2)^n$

(c) A bounded sequence that does not converge to anything. (2 points)

Solution: $a_n = (-1)^n$

2. Suppose that $\{a_n\}_{n=1}^{\infty}$ is a bounded below sequence. Consider the set $A = \{x \mid x = a_n \text{ for some } n\}$. Show that the set A is bounded below and so we can define $B = \inf A = \inf\{a_n\}$ as a real number (and not negative infinity). (2 points)

Solution: Suppose that K is a lower bound for the sequence $\{a_n\}_{n=1}^{\infty}$. Thus $K \leq a_n$ for all n. On the other hand, for each $x \in A$, $x = a_n$ for some n, and so $K \leq x$ as well. Thus K is also a lower bound for A. Hence K is bounded below and we can define B as described above.

Recall the following fact about infimums. If S is a bounded below set and $B = \inf S$, then for every $C > B$, there exists some $x \in S$ such that $x < C$. This is Theorem 1.5.4 from the text for infimums instead of supremums.

3. Suppose next that $\{a_n\}_{n=1}^{\infty}$ is a bounded below sequence that is also non-increasing and that $B = \inf A = \inf\{a_n\}$ as above. Show that $\lim a_n = B$. (4 points)

Hint: Fix $\epsilon > 0$, let $C = B + \epsilon$. Use the hint above to find some $a_n < B + \epsilon$, then use the fact that a_n is non-increasing.

Solution: Following the hint and using the notation from above, for any $\epsilon > 0$, set $C = B + \epsilon$. Since $B = \inf A$, there exists some $x \in A$ such that $x < C = B + \epsilon$ by Theorem 1.5.4 from the text. Since $x \in A$, we know that $x = a_N$ for some integer $N > 0$. Then for any $n > N$, since $\{a_n\}$ is non-increasing, we see that $a_n \leq a_N < C$. Hence $a_n \in [B, C) = [B, B + \epsilon)$, or in other words $|a_n - B| < \epsilon$, as claimed.

4. If $a_1 = 1$ and we recursively define $a_{n+1} = (1 - 2^{-n})a_n$, prove that $\{a_n\}$ converges. (3 points)

Solution: Note that since $2 > 1$, we see that $2^n > 1$ and so $1 > 2^{-n}$ (also note that $2^{-n} > 0$ since $2 > 0$). It follows that $1 > 1 - 2^{-n} > 0$. Multiplying through by a_n we obtain that

$$a_n \geq (1 - 2^{-n})a_n = a_{n+1} \geq 0.$$

Hence $\{a_n\}$ is non-increasing and bounded below, and thus convergent.