WORKSHEET #1 – MATH 3210
FALL 2018

You may work in groups of up to 4. Only one worksheet is required per group.

Definition A *commutative ring* is a set R with a binary operation $+$ (plus) and another binary operation $\cdot$ (times) satisfying the following axioms.

A1 $x + y = y + x$ for all $x, y \in R$.
A2 $x + (y + z) = (x + y) + z$ for all $x, y, z \in R$.
A3 There is an element $0 \in R$ such that $0 + x = x$ for all $x \in R$.
A4 For each $x \in R$, there is an element $-x \in R$ such that $x + (-x) = 0$.

M1 $xy = yx$ for all $x, y \in R$.
M2 $x(yz) = (xy)z$ for all $x, y, z \in R$.
M3 There is an element $1 \in R$ such that $1 \neq 0$ and $1x = x$ for all $x \in R$.
D $x(y + z) = xy + xz$ for all $x, y, z \in R$.

The integers are a ring under the usual $+$ and $\cdot$. Some basic facts about integers immediately follow from the above axioms. For instance, the fact that $0 \cdot x = 0$ for all $x \in R$ can be prove as follows. First note that $0 + 0 = 0$ by A3. Thus

$$x \cdot 0 = x \cdot (0 + 0) = x \cdot 0 + x \cdot 0$$

where the second equality is by item D. We don't know what $x \cdot 0$ is, but if we write $y := x \cdot 0$ then we have just shown that $y = y + y$. But then adding $-y$ to both sides we see that

$$y + (-y) = y + y + (-y)$$

and so

$$0 = y = x \cdot 0 = 0 \cdot x$$

as desired.

Let's prove a similar fact.

1. Suppose that R is a ring and $0' \in R$ is a element such that $0' + y = y$ for all $y \in R$. Prove that $0'$ must be equal to $0 \in R$. (Show the additive inverse is unique)

2. Prove that for every $x \in R$, $(-1)x = -x$. Here -1 is the element corresponding to 1 given by $A4$.

Hint: Add $x = 1 \cdot x$ to $(-1) \cdot x$ and factor (at some step at least).

3. Prove that $(-1) \cdot (-1) = 1$.

Definition A *field* is a ring with the following axiom added.

M4 For each $x \in R$ such that $0 \neq x$, there exists an element (called) $x^{-1} \in R$ such that $x^{-1}x = 1$.

The rational numbers $\mathbb{Q}$ are a field. So are the real numbers, but you might find this less appealing.

4. Suppose that x, y, z are elements of a field R. Show that if $z \neq 0$, then if $xz = yz$, then $x = y$.

5. Suppose that R is a field and $x, y \in R$. Prove that if $xy = 0$ then either $x = 0$ or $y = 0$.

Definition A field F is called an *ordered field* if there is an order relation $\leq$ satisfying the following axioms for all $x, y, z \in F$.

O1 Either $x \leq y$ or $y \leq x$.
O2 If $x \leq y$ and $y \leq x$ then $x = y$.
O3 If $x \leq y$ and $y \leq z$ then $x \leq z$.
O4 If $x \leq y$ then $x + z \leq y + z$.
O5 If $x \leq y$ and $0 \leq z$, then $xz \leq yz$.

6. Suppose that $x, y \in F$ an ordered field. Show that if $0 < x < y$, then $y^{-1} < x^{-1}$.