WORKSHEET #1 – MATH 3210
FALL 2018

You may work in groups of up to 4. Only one worksheet is required per group.

Definition A *commutative ring* is a set R with a binary operation $+$ (plus) and another binary operation $\cdot$ (times) satisfying the following axioms.

- A1 $x + y = y + x$ for all $x, y \in R$.
- A2 $x + (y + z) = (x + y) + z$ for all $x, y, z \in R$.
- A3 There is an element $0 \in R$ such that $0 + x = x$ for all $x \in R$.
- A4 For each $x \in R$, there is an element $-x \in R$ such that $x + (-x) = 0$.
- M1 $xy = yx$ for all $x, y \in R$.
- M2 $x(yz) = (xy)z$ for all $x, y, z \in R$.
- M3 There is an element $1 \in R$ such that $1 \neq 0$ and $1x = x$ for all $x \in R$.
- D $x(y + z) = xy + xz$ for all $x, y, y, z \in R$.

The integers are a ring under the usual $+$ and $\cdot$. Some basic facts about integers immediately follow from the above axioms. For instance, the fact that $0 \cdot x = 0$ for all $x \in R$ can be prove as follows. First note that $0 + 0 = 0$ by A3. Thus

$$x \cdot 0 = x \cdot (0 + 0) = x \cdot 0 + x \cdot 0$$

where the second equality is by item D. We don't know what $x \cdot 0$ is, but if we write $y := x \cdot 0$ then we have just shown that $y = y + y$. But then adding $-y$ to both sides we see that

$$y + (-y) = y + y + (-y)$$

and so

$$0 = y = x \cdot 0 = 0 \cdot x$$

as desired.

Let's prove a similar fact.

1. Suppose that R is a ring and $0' \in R$ is a element such that $0' + y = y$ for all $y \in R$. Prove that $0'$ must be equal to $0 \in R$. (Show the additive inverse is unique)

Solution: Notice that $0' = 0' + 0$ by the defining property of 0, but also $0' + 0 = 0$, by the defining property of $0'$. Hence $0 = 0'$.

2. Prove that for every $x \in R$, $(-1)x = -x$. Here -1 is the element corresponding to 1 given by $A4$.

Hint: Add $x = 1 \cdot x$ to $(-1) \cdot x$ and factor (at some step at least).

Solution: We know that $0 \cdot x = 0$ from the above and we also know that $(1 + -1) = 0$. Hence using property D, and M3, we have

$$0 = 0 \cdot x = (1 + -1) \cdot x = 1 \cdot x + (-1) \cdot x = x + (-1) \cdot x.$$

Using A4 and adding $-x$ to both sides we get

$$-x = x + (-1) \cdot x + (-x) = 0 + (-1) \cdot x = (-1) \cdot x,$$

as desired.

3. Prove that $(-1) \cdot (-1) = 1$.

Solution: We know by **2.** that $z := (-1) \cdot (-1) = -(-1)$. Hence

$$0 = -(-1) + -1 = z + -1.$$

Adding 1 to both sides and using A3, we get

$$1 = 0 + 1 = z + -1 + 1 = z + 0 = z.$$

This completes the proof.

Definition A *field* is a ring with the following axiom added.

M4 For each $x \in R$ such that $0 \neq x$, there exists an element (called) $x^{-1} \in R$ such that $x^{-1}x = 1$.

The rational numbers $\mathbb{Q}$ are a field. So are the real numbers, but you might find this less appealing.

4. Suppose that x, y, z are elements of a field R. Show that if $z \neq 0$, then if $xz = yz$, then $x = y$.

Solution: Since $z \neq 0$, by M4, there exists z^{-1} so that $z^{-1}z = 1$. Then multiplying the equation $xz = yz$ through by z^{-1} and using M3, M2, and M4 we have that

$$x = x \cdot 1 = x(zz^{-1}) = (xz)z^{-1} = (yz)z^{-1} = y(zz^{-1}) = y \cdot 1 = y.$$

5. Suppose that R is a field and $x, y \in R$. Prove that if $xy = 0$ then either $x = 0$ or $y = 0$.

Solution: For a contradiction suppose $x \neq 0$ and $y \neq 0$, then there exists x^{-1} by M4 so that $x^{-1}x = 1$. Then if $xy = 0$, using our assumptions and M4, M1, M2 and the fact that $0 \cdot (\text{anything}) = 0$, we have:

$$0 \neq y = 1 \cdot y = (xx^{-1})y = (x^{-1}xy) = x^{-1}(xy) = x^{-1} \cdot 0 = 0.$$

Definition A field F is called an *ordered field* if there is an order relation $\leq$ satisfying the following axioms for all $x, y, z \in F$.

O1 Either $x \leq y$ or $y \leq x$.
O2 If $x \leq y$ and $y \leq x$ then $x = y$.
O3 If $x \leq y$ and $y \leq z$ then $x \leq z$.
O4 If $x \leq y$ then $x + z \leq y + z$.
O5 If $x \leq y$ and $0 \leq z$, then $xz \leq yz$.

6. Suppose that $x, y \in F$ an ordered field. Show that if $0 < x < y$, then $y^{-1} < x^{-1}$.

Solution: There is more than one correct way to do this. Here's one approach.

First we prove that $1 > 0$. Indeed, if $1 < 0$, then adding -1 to both sides and using O4, we see that $0 < -1$. But then by O5,

$$0 = 0 \cdot (-1) < (-1) \cdot (-1) = 1$$

by **3.** A contradiction. Hence $1 > 0$.

Next we show that $x^{-1} > 0$ since $x > 0$ (and likewise $y^{-1} > 0$). For a contradiction, suppose that $x^{-1} < 0$, then multiplying through by $x > 0$ and using O5, we see that

$$1 = xx^{-1} < 0 \cdot x^{-1} = 0,$$

which contradicts the first thing we did.

Finally, since $x^{-1} > 0$ and $y^{-1} > 0$ and $x < y$, multiplying both sides by $x^{-1}y^{-1}$ we obtain (liberally using O5 and various other properties of being a field, that

$$y^{-1} = y^{-1} \cdot 1 = y^{-1}(x^{-1}x) = (x^{-1}y^{-1})x < (x^{-1}y^{-1})y = x^{-1}(y^{-1}y) = x^{-1} \cdot 1 = x^{-1},$$

as desired.