QUIZ #8 – MATH 3210
FALL 2018

1. This problem has 3 short parts.

(a) Prove that $\frac{d}{dx}(x^3) = 3x^2$. You may either use the limit definition of the derivative or you may use the product rule and the fact that $\frac{d}{dx}x = 1$. (4 points)

Solution: Here's a solution using the product rule.

$$\frac{d}{dx}x^3 = \frac{d}{dx}(x \cdot x^2) = 1 \cdot x^2 + x\frac{d}{dx}x^2 = x^2 + x\frac{d}{dx}(x \cdot x) = x^2 + x(1 \cdot x + x \cdot 1) = x^2 + x^2 + x^2 = 3x^2.$$

Here's a solution using the definition of the derivative.

$$f'(a) = \lim_{x \longrightarrow a} \frac{x^3 - a^3}{x - a} = \lim_{x \longrightarrow a} \frac{(x-a)(x^2 + ax + a^2)}{x - a} = \lim_{x \longrightarrow a} (x^2 + ax + a^2) = a^2 + a^2 + a^2 = 3a^2.$$

(b) Assume that $f(x)$ and $f^{-1}(x)$ are differentiable functions and $f'(x)$ is never zero. Use the chain rule and the identity

$$f(f^{-1}(x)) = x$$

to derive a formula for the derivative of $f^{-1}(x)$. (3 points)

Solution: Taking the derivative of both sides using the chain rule we obtain that:

$$f'(f^{-1}(x))(f^{-1}(x))' = 1.$$

Then solving for $f^{-1}(x)$ we obtain that

$$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}.$$

(c) Use parts (a) and (b) above to show that

$$\frac{d}{dx}(x^{1/3}) = \frac{1}{3}x^{-2/3}.$$

(3 points)

Solution: Let $f(x) = x^3$ so that $f^{-1}(x) = x^{1/3}$. We work on the domain $D = \mathbb{R} \setminus \{0\}$ for $f(x)$, since that is when $f'(x)$ is nonzero (don't take off points for not mentioning that). Note that $f^{-1}(x)$ has the same domain since $f(D) = D$. Then

$$\frac{d}{dx}(x^{1/3}) = (f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))} = \frac{1}{3(f^{-1}(x))^2} = \frac{1}{3x^{2/3}} = \frac{1}{3}x^{-2/3}.$$

2. Suppose that $f : (a, b) \longrightarrow \mathbb{R}$ is differentiable. Further suppose that $f'(x) = 0$ for all $x \in (a, b)$. Prove that f is constant.

Hint: Choose $y, z \in (a, b)$ and assume $y < z$. Apply the mean value theorem to $f : [y, z] \longrightarrow \mathbb{R}$ and use it to show that $f(y) = f(z)$.

Solution: Following the hint, we will be done if we can show that $f(y) = f(z)$ since y and z are chosen arbitrarily. Since f is differentiable on (a, b) it is continuous as well and in particular continuous on $[y, z]$ and differentiable on $(y, z) \subseteq (a, b)$. Thus the Mean Value Theorem says that there exists some $c \in (y, z)$ so that

$$f'(c) = \frac{f(y) - f(z)}{y - z}.$$

But $f'(c) = 0$ since $c \in (y, z) \subseteq (a, b)$ and so $f(y) - f(z) = 0$ and therefore $f(y) = f(z)$.