QUIZ #5 – MATH 3210
FALL 2018

1. Answer the following questions. (2 points each) **(a)** Give a precise definition of what it means that a sequence $\{b_n\}$ has limit equal to b.

Solution: It means that, for every $\varepsilon > 0$, there exists a N so that if $n > N$, then $|b_n - b| < \epsilon$.

(b) The Bolzano-Weierstrass theorem says that "Every bounded sequence has a ..." Fill in the blanks!

Solution: Convergent subsequence.

(c) Suppose that A is a bounded below set. State precisely what it means to take the greatest lower bound of A.

Solution: The greatest lower bound of A is a number L that is both a lower bound for A and bigger than or equal to every other lower bound for A. (Writing this out via symbols is ok too).

(d) Define precisely what it means for a sequence to be Cauchy.

Solution: We say that $\{a_n\}$ is Cauchy if for every $\epsilon > 0$, there exists an N so that if $m, n > N$, then $|a_n - a_m| < \epsilon$.

(e) State precisely what it means for a sequence to be non-increasing.

Solution: A sequence $\{a_n\}$ is non-increasing if for every integer $n \geq 1$, we have that $a_{n+1} \leq a_n$.

2. Prove that the sequence

$$a_n = \frac{(n^2 - 2n + 1)\sin(2^n)}{2n^2 + 3n + \ln(n)}$$

has a convergent subsequence. (10 points)

Hint: What do you need to do in order to apply the Bolzano-Weierstrass theorem? You may use without proof standard facts about sin and ln.

Solution: By the Bolzano-Weierstrass theorem, it is is enough to show that $\{a_n\}$ is bounded. Now

$$|a_n| = \left|\frac{(n^2 - 2n + 1)\sin(2^n)}{2n^2 + 3n + \ln(n)}\right| \leq \frac{|n^2 - 2n + 1| \cdot |\sin(2^n)|}{|2n^2 + 3n + \ln(n)|} \leq \frac{|n^2 - 2n + 1|}{|2n^2 + 3n + \ln(n)|}$$

where the final inequality just comes because $\sin(2^n)$ is bounded between ± 1. We now make two quick observations. Note that $(n^2 - 2n + 1) \leq n^2$ for all $n \geq 1$. Also note that $2n^2 + 3n + \ln(n) \geq 2n^2$. Hence if we make the numerator bigger and the denominator smaller we see that

$$\frac{|n^2 - 2n + 1|}{|2n^2 + 3n + \ln(n)|} \leq \frac{n^2}{2n^2} = 1/2.$$

Putting this all together we have that $|a_n| \leq 1/2$ and hence $\{a_n\}$ is bounded, as desired.