QUIZ #3 – MATH 3210

1. Consider the sequence defined by $a_n = \frac{n-1}{n+1}$. Prove carefully using the and complete sentences definition that

$$\lim a_n = 1.$$

(10 points)

Solution: First, your scratch work might look something like the following:

$$\epsilon > \left|a_n - 1\right| = \left|\frac{n-1}{n+1} - 1\right| = \left|\frac{n-1-(n+1)}{n+1}\right| = \left|\frac{-2}{n+1}\right| = \frac{2}{n+1}.$$

Thus we need $n + 1 > 2/\epsilon$ or in other words, $n > (2/\epsilon) - 1$.

Here is the real proof.

Choose $\epsilon > 0$ and set $N = \lceil(2/\epsilon)\rceil - 1$. Suppose $n > N$, then $n > (2/\epsilon) - 1$ and so $n + 1 > (2/\epsilon)$ and thus

$$\epsilon > \frac{2}{n+1} = \left|\frac{-2}{n+1}\right| = \left|\frac{n-1}{n+1} - 1\right| = \left|a_n - 1\right|.$$

This proves that the sequence a_n has limit 1, as desired.

In this problem, I will step you through a proof that a sequence can't have two different limits.

2. Suppose that $\{a_n\}$ is a sequence and that $\lim a_n = 1$ and also that $\lim a_n = -1$. We are aiming for a contradiction (eventually). You may *not* assume that a_n is a specific example sequence that you like.

(a) Write down the definition of what it means that $\lim a_n = 1$. (2 points)

Hint: For every $\epsilon > 0$.

Solution: For every $\epsilon > 0$, there exists an N_{+1} so that if $n > N_{+1}$, then $|a_n - 1| < \epsilon$.

(b) Likewise write down the definition for what it means that $\lim a_n = -1$ (2 points)

Hint: It should look very similar to what you wrote in (a).

Solution: For every $\epsilon > 0$, there exists an N_{-1} so that if $n > N_{-1}$, then $|a_n + 1| < \epsilon$.

(c) Now, fix $\epsilon = 1$. What you just wrote down in (a) and (b) should tell you that two (possibly different) numbers N, N' exist. Let M be the maximum of these two numbers. Fix $n > M$ and prove that a_n cannot possibly exist that simultaneously satisfies the conditions from parts (a) and (b). This will be our contradiction. (6 points)

Hint: Drawing a picture may help. Make sure to explain why we needed to take the maximum of the two different numbers.

Solution: Choose $\epsilon = 1$. By part (a), we know that there is an N_{+1} so that if $n > N_{+1}$, then $|a_n - 1| < 1$. Likewise by part (b), there is an N_{-1} so that if $n > N_{-1}$ then $|a_n + 1| < 1$.

Fix $M = \max N_{+1}, N_{-1}$. Then if $n > M$, we see that $n > N_{+1}$ and $n > N_{-1}$. Thus we have that *both* $|a_n - 1| < 1$ and $|a_n + 11| < 1$.

We translate these two statements into:

$$\begin{aligned} -1 < \quad & a_n - 1 \quad < 1 \\ -1 < \quad & a_n + 1 \quad < 1 \end{aligned}$$

The first statement implies that $a_n > 0$ (subtracting 1 from both sides). The second implies that $a_n < 0$ (adding one across the inequalities). This is impossible.