QUIZ #10 – MATH 3210
FALL 2018

1. Short answer questions (2 points each)

(a) Suppose $f : (a, b) \longrightarrow \mathbb{R}$ is a function and $c \in (a, b)$. Carefully state what it means for $f'(c) = L$ using the definition of the derivative.

Solution:
It means that

$$\lim_{x \longrightarrow c} \frac{f(x) - f(c)}{x - c} = L,$$

in particular, the limit exists.

(b) Suppose f is differentiable at some a in its domain and $f'(a) \neq 0$. Suppose $b = f(a)$. Write down a formula for $(f^{-1})'(b)$.

Solution: As before, $(f^{-1})'(b) = \frac{1}{f'(a)}$.

(c) Precisely state Cauchy's mean value theorem.

Solution: **Theorem.** Suppose that $f, g : [a, b] \longrightarrow \mathbb{R}$ are continuous and differentiable on (a, b). Further suppose that $g'(x) \neq 0$ for $x \in (a, b)$. Then there exists some $c \in (a, b)$ so that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

(d) Precisely state what it means for f to be integrable on an interval $[a, b]$.

Solution: It means that

$$\inf\{U(f, P) \mid P \text{ a partition of } [a, b]\} = \sup\{L(f, P) \mid P \text{ a partition of } [a, b]\}.$$

(e) Give an example of an integrable function $f : [0, 2] \longrightarrow \mathbb{R}$ which is not continuous and compute $\int_0^2 f(x)dx$.

Solution: The multipart function $f(x) = \begin{cases} 0 & 1 \leq x \leq 2 \\ \pi & 0 \leq x < 1. \end{cases}$ works. Note the integral in my case is just π.

Recall that the first fundamental theorem of calculus says the following.

Theorem. *Suppose* $f : [a, b] \longrightarrow \mathbb{R}$ *is continuous and differentiable on* (a, b)*. Suppose further that* f' *is integrable on* $[a, b]$*. Then*

$$\int_a^b f'(x)dx = f(b) - f(a).$$

2. Prove this theorem. (10 points)

Hint: Take a partition of $[a, b]$, use the Mean Value Theorem to find some appropriate $c_k \in [x_{k-1}, x_k]$ in each subinterval in the partition, and compute the Riemann sum for those c_k values. Make sure to carefully justify each step in your proof using words and complete sentences where appropriate.

Solution: Following the hint, by the Mean Value Theorem we may choose $c_k \in (x_{k-1}, x_k)$ so that $f'(c_k) = \frac{f(x_k)-f(x_{k-1})}{x_k-x_{k-1}}$. It follows that the Riemann sum for these values is

$$\sum_{k=1}^{n} f'(c_k)(x_k - x_{k-1}) = \sum_{k=1}^{n} f(x_k) - f(x_{k-1}) = f(b) - f(a).$$

Thus for any partition, $L(f, P) \leq f(b) - f(a) \leq U(f, P)$ and hence

$$\underline{\int_a^b} f'(x)dx = \sup_P L(f, P) \leq f(b) - f(a) \leq \inf U(f, P) = \overline{\int_a^b} f'(x)dx$$

Since f' is integrable, the left and right side of the inequalities above are equal and so

$$\int_a^b f'(x)dx = f(b) - f(a).$$