QUIZ #1 – MATH 3210
FALL 2018

1. Consider the following sets.

$$A = \{7, 3, \emptyset, \star\}, B = \mathbb{Z}, C = [0, \pi].$$

Compute the following:

(a) $A \cap B \cap C$. (3 points)

Solution: $A \cap B = \{7, 3\}$ and so

$$A \cap B \cap C = \{7, 3\} \cap [0, \pi] = \{3\}.$$

(b) $A \cup (B \cap C)$. (3 points)

Solution: $B \cap C = \{0, 1, 2, 3\}$. Thus

$$A \cup (B \cap C) = A \cup \{0, 1, 2, 3\} = \{7, 3, \emptyset, \star, 0, 1, 2\}.$$

Now suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is the function $f(x) = 3x - 1$.

(c) What is $f^{-1}(A \cap B)$? (3 points)

Solution: First note $A \cap B = \{7, 3\}$ from (a). We need to find x so that $f(x) \in A \cap B$. Suppose first $f(x) = 7$, then $3x - 1 = 7$ and so $x = 8/3$. Next if $f(x) = 3$, then $3x - 1 = 3$ so $x = 4/3$. Thus,

$$f^{-1}(A \cap B) = f^{-1}(\{7, 3\}) = \{8/3, 4/3\}.$$

(d) What is $f(C \setminus A)$? (3 points)

Hint: The $\setminus$ notation means "set minus" and in some other texts is written as $C - A$.

Solution: First we compute $C \setminus A$. This is just the union of two intervals, $[0, 3) \cup (3, \pi]$. Now we need to compute the image of this set. Our function is linear, so it will send intervals to intervals. Note $f([0, 3)) = [-1, 8)$ and $f((3, \pi]) = (8, 3\pi - 1]$. Putting this together, we see that

$$f(C \setminus A) = [-1, 8) \cup (8, 3\pi - 1].$$

See the back page for the second problem.

2. Prove by induction that every number of the form $5^n - 2^n$ with $n \in \mathbb{N}$, is divisible by 3. (8 points)

Solution: We begin with the base case of $n = 1$. In this case $5^1 - 2^1 = 3$ which is divisible by 3, and our base case is done.

Next suppose that $5^k - 2^k$ is divisible by 3. We need to prove that $5^{k+1} - 2^{k+1}$ is also divisible by 3. We write

$$5^{k+1} - 2^{k+1} = 5 \cdot 5^k - 2 \cdot 2^k,$$

but this is *not* clearly a multiple of $5^k - 2^k$, so we must do something else. Using the fact that $5 = 3 + 2$ we have that:

$$5 \cdot 5^k - 2 \cdot 2^k = (3 + 2) \cdot 5^k - 2 \cdot 2^k = 3 \cdot 5^k - 2 \cdot 5^k - 2 \cdot 2^k = 3 \cdot 5^k - 2 \cdot (5^k - 2^k).$$

Now, $3 \cdot 5^k$ is divisible by 3 certainly. Also, $2 \cdot (5^k - 2^k)$ is divisible by 3 by our inductive hypothesis. Thus their difference is also divisible by 3. But now we have shown that

$$5^{k+1} - 2^{k+1} = 3 \cdot 5^k - 2 \cdot (5^k - 2^k)$$

is divisible by 3 which completes the proof by induction that $5^n - 2^n$ is divisible by 3 for every natural number n. Notice this problem is also an example in the text, so you can see a slightly different proof there.