MATH 3210 – MIDTERM #2

Your Name

- You have 50 minutes to do this exam.
- No calculators or notes!
- For justifications, please use complete sentences and make sure to explain any steps which are questionable.
- Good luck!

Problem	Total Points	Score
1	24	
2	26	
3	25	
4	25	
Total	100	

1. Short answer questions (3 points each).

(a) State the second fundamental theorem of calculus.

Solution: Suppose that $f : [a, b] \longrightarrow \mathbb{R}$ is integrable, then the function $F(x) = \int_a^x f(t)dt$ is continuous and furthermore that F is differentiable with $F'(x) = f(x)$ if f is continuous at x.

(b) Precisely define what it means for a series $\sum_{k=1}^{\infty} a_k$ to converge.

Solution: It means that the sequence of partial sums $\sum_{k=1}^{n} a_k$ converges.

(c) Compute $\lim_{x \longrightarrow 0^-} \dfrac{\cos(x) - 1}{\ln(x) - x}$ if it exists. If it does not exist, say that.

Solution: It doesn't exist, $\ln(x)$ isn't defined for negative x.

(d) Precisely define the function $\ln(x)$ using a definite integral where x is one of the bounds.

Solution: $\ln(x) = \int_1^x \frac{1}{t} dt$.

(e) Identify the critical points of the function $f : [-2\pi, 2\pi] \longrightarrow \mathbb{R}$ defined by $f(x) = \sin(|x - 1|)$.

Solution: The end points are critical points. $x = \pm 2\pi$. The function is also not differentiable at $x = 1$, so that is a critical point. Finally, the function has zero derivative at $x = 1 - 3\pi/2, 1 - \pi/2, 1 + \pi/2, 1 + 3\pi/2$.

(f) Give a precise definition of the statement $\lim_{x \longrightarrow -\infty} f(x) = L$.

Solution: It means that for every $\epsilon > 0$, there exists a $K \in \mathbb{R}$, so that if $x < K$ then $|x - L| < \epsilon$

(g) Consider $f : [-1, 2] \longrightarrow \mathbb{R}$ defined by $f(x) = x^2$. Let $P = \{-1, 0, 1, 2\}$. Compute $L(f, P)$.

Solution: We have $m_0 = 0$ (the min of x^2 on $[-1, 0]$ is 0), $m_1 = 0$ and $m_2 = 1$. Hence $L(f, P) = 0 + 0 + 1$.

(h) Suppose that $P \subseteq Q$ is a refinement of partitions of the interval $[0, 1]$. Consider the function $f(x) = \ln(2 + \cos(e^x)) - \sqrt{x^2 + \frac{1}{x^2+1}}$. Arrange the following values from smallest to largest.

$$U(f, P), U(f, Q), L(f, P), L(f, Q), \int_0^1 f(x)dx.$$

Solution: $L(f, P) \leq L(f, Q) \leq \int_0^1 f(x)dx \leq U(f, Q) \leq U(f, P)$

2. Consider the function $f : (-1, \infty) \longrightarrow \mathbb{R}$ defined by

$$f(x) = \int_{-x}^{\ln(x+e)} (2t)e^{t^2}\,dt.$$

(a) Compute $f'(x)$. (16 points)

Solution: We break this up into two integrals, $\int_0^{\ln(x+1)}(2t)e^{t^2}dt$ and $\int_{-x}^{0}(2t)e^{t^2}dt$. Using the chain rule, we get for the first integral

$$(2\ln(x+e))e^{(\ln(x+e))^2}\frac{1}{x+e}.$$

For the second (switching the integrals, and doing the chain rule, and plugging in $-x$), we get

$$(-1)(-1)(-1)2xe^{x^2}.$$

Summing this we get

$$(2\ln(x+e))e^{(\ln(x+e))^2}\frac{1}{x+e} - 2xe^{x^2}.$$

(b) Verify that $f(0) = e - 1$. (4 points)

Hint: Use u substitution if you can't eyeball an anti-derivative.

Solution: An anti-derivative is e^{t^2}, so we have that the integral is $\int_0^1(2t)e^{t^2}dt = e^{1^2} - e^0 = e - 1$.

(c) Compute $(f^{-1})'(e-1)$. (6 points)

Solution: Since $f(0) = e - 1$, we certainly have that $f^{-1}(e-1) = 0$ (it could be based on what we know that $f(y) = e - 1$ for some other y, but lets not worry about that, it doesn't happen near 0 anyways.) Thus $(f^{-1})'(e-1) = \frac{1}{f'(f^{-1}(e-1))} = \frac{1}{f'(0)}$. A quick computation show that $f'(0) = 2$, thus the answer is $1/2$.

3. Suppose that $(a, b) \subseteq \mathbb{R}$ is a non-empty open interval and that we have a differentiable function $f : (a, b) \longrightarrow \mathbb{R}$ such that $f'(x) < 0$ for all $x \in (a, b)$. Use the Mean Value Theorem to prove that f is strictly decreasing. (25 points)

Solution: Choose $x < y$ with $x, y \in (a, b)$. Then there exists $c \in (x, y)$ such that $f'(c) = \frac{f(y)-f(x)}{y-x}$. Since $f'(c) < 0$, by clearing denominators we see that $f(y) - f(x) < 0$. Thus $f(y) < f(x)$ and so f is strictly decreasing.

4. Prove the following theorem.

Theorem. *Suppose $a < b$ are real numbers. Suppose $f : [a, b] \longrightarrow \mathbb{R}$ is continuous. Prove that*

$$\int_a^b f(x)dx$$

exists (in other words, that f is integrable on $[a, b]$).

Hint: Break up $[a, b]$ into n equal subintervals where n is small enough so that you can force M_k and m_k close really close to each other by *uniform* continuity.

Solution: Fix $\epsilon > 0$. We will show that $U(f, P) - L(f, P) < \epsilon$ for some partition P. First, since f is continuous on $[a, b]$, it is uniformly continuous, so there exists $\delta > 0$ so that if $|x - y| < \delta$ for some $x, y \in [a, b]$, then $|f(x) - f(y)| < \epsilon/(b - a)$. Now, create a partition P by dividing $[a, b]$ into m equal subintervals of length less than δ. Thus on each interval, $[x_{k-1}, x_k]$, we have by the extreme value theorem since f is continuous, that $f([x_k, x_{k+1}])$ has a maximum $M_k = f(y_k)$ and a minimum $m_k = f(z_k)$, with $y_k, z_k \in [x_{k-1}, x_k]$. By construction, $|M_k - m_k| = |f(y_k) - f(z_k)| < \epsilon/(b - a)$. Therefore,

$$U(f, P) - L(f, P) = \sum_{k=1}^{m} (M_k - m_k) \frac{b - a}{m} < m \cdot \frac{\epsilon}{b - a} \cdot \frac{b - a}{m} = \epsilon,$$

as desired.

(EC) Suppose $h : [a, b) \longrightarrow \mathbb{R}$ is a continuous function which is differentiable on (a, b) and where $\lim\limits_{x \longrightarrow b^-} h(x) = \infty$. Suppose further that $h([a, b)) = [c, \infty)$. Suppose that $f : [c, \infty) \longrightarrow \mathbb{R}$ is integrable on every closed interval $[c, r] \subseteq [c, \infty)$. Define a new function $L : [a, b] \longrightarrow \mathbb{R}$ by

$$L(t) = \begin{cases} f(h(t))h'(t), & t \in [a, b) \\ 7, & t = b \end{cases}$$

and suppose it is integrable on $[a, b]$. Prove that

$$\int_{h(a)}^{\infty} f(x)dx = \int_a^b L(t)dt.$$

In particular, show that the improper integral on the left exists.

Solution: L is also integrable on each $[a, r]$ for $r < b$. Hence consider the function $H(x) = \int_a^x L(t)dt$. We know from the fundamental theorem of calculus part II that $H(x)$ is continuous on $[a, b]$ so that $\lim_{x \longrightarrow b} H(x) = H(b)$. Note $H(x) = \int_{h(a)}^{h(x)} f(u)du$ by u-substitution. Therefore

$$\int_a^b L(t)dt = H(b) = \lim_{x \longrightarrow b} \int_{h(a)}^{h(x)} f(u)du = \int_{h(a)}^{\infty} f(u)du.$$