MATH 3210 – MIDTERM #1 SOLUTIONS

1. Short answer questions (3 points each).

(a) What is the completeness axiom of the real numbers? (As we defined it in class).

Solution: Every bounded above set of real numbers has a least upper bound.

(b) Give a precise definition about what it means for a sequence $\{a_n\}$ to converge to $+\infty$.

Solution: For every real number K, there exists a real number $N > 0$ so that if $n > N$, then $a_n > K$.

(c) Give an example of a bounded function $f : \mathbb{R} \longrightarrow \mathbb{R}$ that is not continuous.

Solution: $f(x) = \begin{cases} 1 & x \geq 0 \\ 0x < 0 & \end{cases}$

(d) Precisely define what it means for a sequence of functions $\{f_n : D \longrightarrow \mathbb{R}\}$ to be uniformly convergent to a function $f : D \longrightarrow \mathbb{R}$.

Solution: It means for every $\epsilon > 0$ there exists an $N > 0$ so that if $n > N$, we have that $|f_n(x) - f(x)| < \epsilon$ for all $x \in D$.

(e) Give an example of a domain D and a function $g : D \longrightarrow \mathbb{R}$ so that g is not uniformly continuous.

Solution: You can take $D = \mathbb{R}$ and $g(x) = x^2$. Or you can take $D = (0, \infty)$ and $g(x) = 1/x$. Many other things work too.

(f) What does the Bolzano-Weierstrass theorem say?

Solution: It says that every bounded sequence of real numbers has a convergent subsequence.

(g) Give an example of a monotone sequence that is not convergent.

Solution: $a_n = n$.

(h) Suppose that $\{a_n\}$ is a sequence such that $\lim a_n = 11$. Compute $\limsup(\frac{a_n}{n})$.

Solution: The answer is zero. In fact, $\lim \frac{a_n}{n} = 0$ since a_n converges to a finite limit and the denominator n converges to ∞.

2. Consider the function $f(x) = x^3 + 1$. Prove directly from the definition that $f(x)$ is continuous at $a = 2$.

(26 points) *Hint:* You may use that $u^3 - v^3 = (u - v)(u^2 + uv + v^2)$.

Solution: Note first that $f(a) = 2^3+1 = 9$. Choose $\epsilon > 0$ and set $\delta = \min(1, \epsilon/19)$. Suppose that $|x-a| = |x-2| < \delta$. Then since $\delta \leq 1$, we have $x \in (1, 3)$. We see that then $x^2+2x+4 < 9+6+4 = 19$ for such x. We have

$$|f(x) - 9| = |x^3 + 1 - 9| = |x^3 - 8| = |x - 2| \cdot |x^2 + 2x + 4| < |x - 2| \cdot 19 < \delta \cdot 19 \leq (\epsilon/19) \cdot 19 = \epsilon.$$

This completes the proof.

3. Suppose that $\{a_n\}$ and $\{b_n\}$ are two convergent sequences that both converge to the same $L \in \mathbb{R}$. Define a new sequence c_n as follows:

$$c_n = \begin{cases} a_n & \text{if } n \text{ is odd} \\ b_n & \text{if } n \text{ is even} \end{cases}$$

In other words $\{c_n\}$ is the sequence

$$\{a_1, b_2, a_3, b_4, a_5, b_6, a_7, b_8, \dots\}.$$

Prove that c_n also converges to L. (25 points)

Solution: Choose $\epsilon > 0$. Since $a_n \longrightarrow L$, there exists an N_1 so that if $n > N_1$, then $|a_n - L| < \epsilon$ for $n > N_1$. Since $b_n \longrightarrow L$ there exists an N_2 so that if $n > N_2$, then $|b_n - L|\epsilon$. Now pick $N = \max(N_1, N_2)$ and suppose that $n > N$ so that $n > N_1$ and $n > N_2$.

Consider $|c_n - L|$. If n is odd, then $|c_n - L| = |a_n - L| < \epsilon$. On the other hand if n is even then $|c_n - L| = |b_n - L| < \epsilon$. In either case, $|c_n - L| < \epsilon$ and the proof is complete.

4. Suppose that I is a closed and bounded interval and that $f : I \longrightarrow \mathbb{R}$ is a continuous function. Prove that f is uniformly continuous. (25 points)

Solution: See the text.

(EC) Consider a closed and bounded interval $I = [a, b]$. We make the following definition.

Definition. An *open cover by intervals* of I is a set of open intervals $U_t \subseteq \mathbb{R}$ so that $I \subseteq \bigcup_t U_t$.

Now suppose that $\{U_t\}$ is open cover by intervals of I and that there are infinitely many U_t. Prove that there is a finite collection $U_{t_1}, \dots, U_{t_n}$ of open intervals from our cover so that

$$I \subseteq U_{t_1} \cup \dots \cup U_{t_n}. \text{ (10 points)}$$

This is called showing that I has a *finite subcover.*

Hint: Consider the set S of $c \in [a, b] = I$ such that $[a, c]$ has a finite subcover among the $\{U_t\}$. Take the supremum of S, show it must equal b and think about what this says about $[a, b)$.

Solution: First note that the set S is nonempty since $[a, a]$ has a finite subcover (take any U_t containing a). Thus $a \in S$. Following the hint, let $s = \sup S$. Since $S \subseteq [a, b]$, $s = \sup S \leq \sup[a, b] = b$ and so $s \in [a, b]$ (since s is an upper bound for $a \in S$). Thus there exists some $U_t = (u_t, v_t)$ in our cover containing s. Since $u_t < s$, by the definition of supremum there exists some interval $[a, c]$ with $u_t < c < s$ with $[a, c]$ having a finite subcover $\{U_{t_1}, \dots, U_{t_n}\}$. Then $\{U_t, U_{t_1}, \dots, U_{t_n}\}$ is a finite subcover of $[a, s]$, and so if $s = b$ we are done. Furthermore, if $s < b$ let $v = \min((v_t + s)/2, b)$, then $\{U_t, U_{t_1}, \dots, U_{t_n}\}$ is a finite subcover of $[a, v]$. If $v = b$ we are done. If $v < b$, then $v = (s + v_t)/2 \in S$ which contradicts the fact that $s = \sup S$ since $v_t > s$ and so $(v_t + s)/2 > s$.