EXTRA CREDIT #2 – MATH 3210
FALL 2018

DUE WEDNESDAY, NOVEMBER 28TH

You are allowed to work with other people from this class. However, each person must turn in their own extra credit assignment in order to get credit. The points on this assignment will be directly added to your homework/quiz/worksheet score.

Let X be a set (such as the real numbers). Consider a collection T of subsets of X that satisfy the following properties.

(i) $\emptyset \in T$.
(ii) $X \in T$.
(iii) If $U, V \in T$ then $U \cap V \in T$.
(iv) If $\{U_i\} \subseteq T$ then $\bigcup U_i \subseteq T$ (there might be infinitely many or even uncountably many U_i).

Such a set T is called a *topology for* X and the sets $U \in T$ are called *open*.

1. Consider X to be the real numbers and define T to be the collection of open[1] subsets of $\mathbb{R}$. Show T satisfies properties (i) through (iv) above. In other words, show it is a topology for $\mathbb{R}$. It is called the *Euclidean topology for* $\mathbb{R}$. (4 points).

Definition. Suppose X and Y are sets and S is a topology for X and T is a topology for Y. We say that a function $f : X \longrightarrow Y$ is continuous if for each $U \in T$, we have that $f^{-1}(U) \in S$.

2. Consider the set $X = \{A, B, C\}$ with topology $S = \{\emptyset, \{A\}, \{A, B\}, \{A, B, C\}\}$. Furthermore let $Y = \mathbb{R}$ with the Euclidean topology.

(a) Show that the function $f : X \longrightarrow Y$ defined by $f(x) = 7$ is continuous. (2 points)

(b) Show that the function $f : X \longrightarrow Y$ defined by $f(A) = 0$, $f(B) = 1$, $f(C) = 2$ is not continuous. (2 points)

(c) Show that if $f : X \longrightarrow Y$ is continuous, then f must be a constant function $f(x) = K$ for some $K \in \mathbb{R}$. (2 points)

Hint: Think about $f(C)$. What are the open sets in S containing C?

3. Define a new topology S on $\mathbb{R}$ by declaring a set $U \subseteq \mathbb{R}$ to be in S if $\mathbb{R} \setminus U$ is finite. Show the following:

(a) Show that the function $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^2$ is continuous if both the source and target of f have the new S topology. (2 points)

(b) Let T denote the Euclidean topology on $\mathbb{R}$. Consider the identity function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x$. Is f continuous if the source has topology T (from **(a)**) and the target has topology S? What if the source has topology S and the target has topology T. Prove your answers. (4 points)

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[1]Recall from a previous extra credit that a subset $U \subseteq \mathbb{R}$ is open if for each $x \in U$, there exists a δ so that $(x - \delta, x + \delta) \subseteq U$.

4. Consider a subset $Y \subseteq \mathbb{R}$. Define a topology T on Y as follows. $U \in T$ if there exists an open set $V \subseteq \mathbb{R}$ (in the Euclidean topology) so that $U = V \cap Y$. Show that T is a topology for Y. It is called the *subspace topology for* Y. (4 points).

Definition. Suppose that X has a topology T. We say that X is *disconnected* if there are two non-empty open sets $U, V \in T$ such that

- $U \cap V = \emptyset$.
- $U \cup V = X$.

If X is not disconnected, then we say that X is *connected*.

5. Show that any closed interval $I \subset \mathbb{R}$ is *connected*. Here we use the Euclidean topology on $\mathbb{R}$ and the subspace topology T on I. (4 points).

Hint: Suppose that $I = [a, b]$ is not connected. Thus there exists $U, V \in T$ so that $U \cap V = \emptyset$ and $U \cup V = I$. Suppose without loss of generality that $a \in U$. Let $c = \inf(I \cap V)$. Since $c \in I$, we have $c \in U$ or $c \in V$. Show that either way yields a contradiction.

6. Consider the set $C = [0, 1] \cap \mathbb{Q}$ (the rational numbers between 0 and 1). Give $C \subseteq \mathbb{R}$ the subspace topology. Prove that C is disconnected. (3 points).

7. Suppose that $f : X \longrightarrow Y$ is a surjective map between sets. Suppose that X has topology S and Y has topology T and that f is continuous with respect to these topologies. Suppose that X is connected, prove that Y is also connected. (4 points)

8. Provide a new proof of the intermediate value theorem as follows. Let I be a closed interval and that $f : I \longrightarrow \mathbb{R}$ is continuous. First show that $f : I \longrightarrow f(I)$ is also continuous (where $f(I) \subseteq \mathbb{R}$ has the subspace topology). Suppose that the conclusion of the intermediate value theorem does not hold and derive a contradiction using **5.** and **7.**. (5 points)