WORKSHEET #4 – MATH 311W

OCTOBER 19TH, 2012

We will study functions.

1. Suppose that $f : X \to Y$ and $g : Y \to Z$ are injective functions. Prove that $g \circ f$ is injective.

Solution: Suppose that $a, b \in X$ and that $g \circ f(a) = g \circ f(b)$. We want to prove that $a = b$. We know $g(f(a)) = g(f(b))$. Since g is injective, we know that $f(a) = f(b)$. Since f is injective, we know that $a = b$. This completes the proof.

2. Suppose that $f : X \to Y$ and $g : Y \to Z$ are surjective functions. Prove that $g \circ f$ is surjective.

Solution: Choose $a \in Z$. We want to find $c \in X$ such that $g \circ f(c) = a$. Since g is surjective, there exists $b \in Y$ such that $g(b) = a$. Since f is surjective, there exists $c \in X$ such that $f(c) = b$. Thus $g \circ f(c) = g(f(c)) = g(b) = a$ as desired.

3. Suppose that $f : X \to Y$ and $g : Y \to Z$ are functions and that $g \circ f$ is surjective. Prove that g is surjective.

Solution: Choose $z \in Z$. We want to find $y \in Y$ such that $g(y) = z$. Since $g \circ f$ is surjective, there exists $x \in X$ such that $g(f(x)) = g \circ f(x) = z$. Set $y = f(x)$. Then $z = g(f(x)) = g(y)$ as desired.

4. Suppose that $f : X \to Y$ and $g : Y \to Z$ are functions and that $g \circ f$ is injective. Prove that f is injective.

Solution: Choose $a, b \in X$ and suppose that $f(x) = f(y)$. We want to show that $x = y$. By applying g to both sides of the equation $f(x) = f(y)$ we obtain $g \circ f(x) = g(f(x)) = g(f(y)) = g \circ f(y)$. Since $g \circ f$ is injective, we obtain that $x = y$ as desired.

5. Suppose that $f : X \to Y$ and $g : Y \to Z$ are functions and that $g \circ f$ is injective. Give an example that proves that g is not necessarily injective.

Solution: Consider the function $f : \{1\} \to \{2, 3\}$ which sends 1 to 2. Consider the function $g : \{2, 3\} \to \{4\}$ which sends both 2 and 3 to 4. Then consider $g \circ f : \{1\} \to \{4\}$. This function sends 1 to 4 (by first sending 1 to 2 and then 2 to 4). This function is bijective however, g certainly is not injective since 2 and 3 get sent to the same thing.

6. Suppose that $f : X \to Y$ and $g : Y \to Z$ are functions and that $g \circ f$ is surjective. Give an example that proves that f is not necessarily injective.

Solution: Consider the function $f : \{0, 1\} \to \{2, 3\}$ which sends both 0 and 1 to 2. Consider the function $g : \{2, 3\} \to \{4\}$ which sends both 2 and 3 to 4. Then consider $g \circ f : \{0, 1\} \to \{4\}$. This function sends both 0 and 1 to 4. It is surjective. However, f is neither injective or surjective!