

**WORKSHEET #3 – MATH 2200  
SPRING 2018**

NOT DUE

This is very much like what the first page of the exam will be.

1. Short answer questions.

(a) Give an example of a surjective function but non-injective function  $f : \mathbb{Z} \rightarrow \mathbb{Z}$ .

**Solution:** There are lots of correct answers. One such answer is

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ x + 1 & \text{if } x < 0 \end{cases}$$

(b) What does it mean for a relation on a set to be reflexive?

**Solution:** If the set is  $A$ , it means that  $a \sim a$  for all  $a \in A$ , or in other words,  $(a, a)$  is in the relation.

(c) Is the proposition “If  $1 + 2 = 2$  then  $3 + 3 = 7$ ” true?

**Solution:** Yes.

(d) What is the power set of  $\{\emptyset\}$ ?

**Solution:**  $\{\emptyset, \{\emptyset\}\}$ .

(e) Is  $\neg(q \vee q) \leftrightarrow \neg q \wedge \neg p$  a tautology?

**Solution:** No.

(f) Consider the function  $g : \{1, 2\} \rightarrow \mathbb{R}$  which is defined by  $g(x) = x^3$ . Is  $g$  surjective? Is  $g$  injective?

**Solution:** Not surjective. It is injective though.

(g) If  $S$  is a set, is it always true that  $\emptyset \subseteq S$ ?

**Solution:** Yes.

2. Short answer questions continued.

(a) Give an example of an uncountable set.

**Solution:**  $\mathbb{R}$  (or  $\mathbb{R}_{>0}$ , or  $[0, 1)$ , or  $\mathcal{P}(\mathbb{Z})$ , or many others too).

(b) Is  $\mathbb{Z} \times \{1, 2, 3\}$  countable?

**Solution:** Yes.

(c) Consider the proposition  $\exists x \forall y (Q(x, y) \rightarrow P(x, y))$ . Express the negation of the proposition in such a way that there is no negation sign outside of a quantifier, or outside of parentheses (not counting the parentheses  $Q(x, y)$  or  $P(x, y)$ ).

**Solution:**  $\forall x \exists y (\neg P(x, y) \wedge Q(x, y))$ .

(d) Is it always true that  $S \neq \mathcal{P}(S)$ ?

**Solution:** Yes. Those two sets are never equal.

(e) How many injective functions are there from  $A = \{1\}$  to  $B = \{2, 3, 4\}$ ?

**Solution:** 3, we can send 1 to any of three different elements. Since there's only one element in the domain, all such functions are injective.

(f) Do the even integers have the same cardinality as the rational numbers?

**Solution:** Yes.

(g) Give an example of a relation on the set  $A = \{1, 2, 3\}$  that is not transitive.

**Solution:** Consider  $\{(1, 2), (2, 3)\}$ . Note  $1 \sim 2$  and  $2 \sim 3$  but 1 and 3 are not related.

(h) Is the assertion  $\{1, 2\} \in \{\emptyset, 1, 2\}$  true?

**Solution:** No.

Here's a couple problems on sets that are similar to what you might see on the exam.

3. Suppose that  $A, B$  and  $C$  are sets. Prove carefully (using complete sentences) that

$$(A \cap B) \cup (A \cup B^c)^c = B$$

Recall that  $S^c$  denotes the complement of a set  $S$ .

**Solution:** We notice that  $(A \cup B^c)^c = A^c \cap B$  and so it suffices to show that

$$(A \cap B) \cup (A^c \cap B) = B.$$

We first show  $\subseteq$ . Suppose that  $x \in (A \cap B) \cup (A^c \cap B)$ . Thus  $x \in A \cap B \subseteq B$  or  $x \in A^c \cap B \subseteq B$ . However, in either case  $x \in B$  and so  $\subseteq$  is proven.

For the reverse inclusion  $\supseteq$  suppose that  $x \in B$ . There are two possibilities, either  $x \in A$  or  $x \in A^c$ . If  $x \in A$  then  $x \in A \cap B \subseteq (A \cap B) \cup (A^c \cap B)$ . If  $x \in A^c$ , then  $x \in A^c \cap B \subseteq (A \cap B) \cup (A^c \cap B)$ . In either case  $x \in (A \cap B) \cup (A^c \cap B)$  and so we have proven  $\supseteq$ .

Since we proved that both  $\subseteq$  and  $\supseteq$  hold, we are done.

**Solution: (Alternate complicated version)** We first show  $\subseteq$ . Suppose  $x \in (A \cap B) \cup (A \cup B^c)^c$ . There are two possibilities, either  $x \in A \cap B$  or  $x \in (A \cup B^c)^c$ . In the former case, since  $A \cap B \subseteq B$ , we see that  $x \in B$ . Next we assume that  $x \in (A \cup B^c)^c$ . Thus  $x \notin A$  and  $x \notin B^c$ , the second of which implies that  $x \in B$ . Thus either way,  $x \in B$ . This proves  $\subseteq$ .

Next we show  $\supseteq$ . We proceed by contradiction so suppose that  $x \in B$  and  $x \notin (A \cap B) \cup (A \cup B^c)^c$ . From the latter condition, we see that  $x \notin (A \cap B)$  and  $x \notin (A \cup B^c)^c$ . Since  $x \notin (A \cap B)$ , we see that  $x \notin A$  or  $x \notin B$ , but of course we assumed that  $x \in B$  and so we must have  $x \notin A$ . From the fact that  $x \notin (A \cup B^c)^c$  we see that  $x \in (A \cup B^c)$  and so  $x \in A$  or  $x \in B^c$ . This implies that  $x \in A$  or  $x \notin B$ , but we assumed that  $x \in B$ , and so we must have  $x \in A$ . Putting this together we see that.

- $x \notin A$  and
- $x \in A$ .

This is clearly impossible, which is a contradiction and proves  $\subseteq$ .

4. Suppose that  $A, B$  and  $C$  are sets and that  $A \neq \emptyset$ . Prove that  $B = C$  if and only if  $A \times B = A \times C$ .

**Solution:** Suppose first that  $B = C$ , then clearly  $A \times B = A \times C$ , so the  $\rightarrow$  direction is clear.

For the reverse direction, suppose that  $A \times B = A \times C$ . Now we show that  $B = C$ . Choose first  $b \in B$ . Since  $A \neq \emptyset$  we may pick  $a \in A$ . Since  $b \in B$ , we see that  $(a, b) \in A \times B = A \times C$ . Thus  $b \in C$  and we have proved that  $B \subseteq C$ . By symmetry,  $C \subseteq B$ .

This completes the proof.