# WORKSHEET #2 – MATH 2200
# SPRING 2018

NOT DUE, JUST PRACTICE

**1.** Suppose that $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are injective functions. Prove that $g \circ f$ is also injective.

**Solution:** Choose $a, a' \in A$ and suppose that $g(f(a)) = (g \circ f)(a) = (g \circ f)(a') = g(f(a'))$. We will show that $a = a'$ which will prove that $g \circ f$ is injective. Now, because $g$ is injective, for any $x, x' \in B$, if $g(x) = g(x')$ then $x = x'$. In our case we have $g(f(a)) = g(f(a'))$. Setting $x = f(a)$ and $x' = f(a')$, we see that $f(a) = f(a')$. But now since $f$ is injective, we see that $a = a'$. This proves that $g \circ f$ is injective, as desired.

**2.** Is the set of integers divisible by 5 but not by 7, (a) finite, (b) countably infinite, or (c) uncountable?

**Solution:** Countably infinite.

Remember, to show a set $S$ is countable it suffices to show that there is a surjective map $f : \mathbb{Z}_{>0} \longrightarrow S$, or in other words it suffices to show that there is a way to list/enumerate them, so that all of the elements of $S$ appear in the list. To show something is countably infinite, we must show it is countable and infinite.

Now, we can list the positive elements of our set in an infinite list $(7, 14, 21, 28, 42, \dots)$ and likewise the negative ones as well $(-7, -14, -21, -28, -42, \dots)$. To list all of them in a list we can just alternate:

$$(7, -7, 14, -14, 21, -21, 28, -28, 42, -42, \dots).$$

**3.** Suppose that $A, B$ are two countable infinite sets. Show that $A \cup B$ and $A \times B$ are also countable.

**Solution:** First we write $A = \{a_1, a_2, a_3, a_4, \dots\}$ and $B = \{b_1, b_2, b_3, b_4, \dots\}$. Consider the function $f : \mathbb{Z}_{>0} \longrightarrow A \cup B$ which sends

$$n \mapsto \begin{cases} a_{n/2} & \text{if } n \text{ is even} \\ b_{(n+1)/2} & \text{if } n \text{ is odd} \end{cases}$$

This function is surjective and it follows immediately that the cardinality of $A \cup B$ is $\leq$ to that of $\mathbb{Z}_{>0}$, or in other words, $A \cup B$ is countable.

For $A \times B$, you use the same pattern that we used to show that the positive rational numbers are countable. We simply apply it to:

$$\begin{array}{ccccc} (a_1, b_1) & (a_1, b_2) & (a_1, b_3) & (a_1, b_4) & \dots \\ (a_2, b_1) & (a_2, b_2) & (a_2, b_3) & (a_2, b_4) & \dots \\ (a_3, b_1) & (a_3, b_2) & (a_3, b_3) & (a_3, b_4) & \dots \\ (a_4, b_1) & (a_4, b_2) & (a_4, b_3) & (a_4, b_4) & \dots \\ \dots & \dots & \dots & \dots & \end{array}$$

**4.** Suppose that $A$ is an uncountable set. If $B \subseteq A$ is countably infinite, prove that $A - B$ is uncountable.

**Solution:** Suppose for a contradiction that $C = A - B$ was countable. Then $B \cup C$ would be countable by the previous problem. However, $B \cup C = B \cup (A - B) = A$, and hence $A$ would be countable, contradicting our hypothesis. This completes the proof.

**5.** Give an example of two uncountable sets $A$ and $B$ so that

(a) $A - B$ is finite.

(b) $A - B$ is countably infinite.

(c) $A - B$ is uncountable.

**Solution:** For (a), you could let $A = \mathbb{R}$ and $B = \mathbb{R} - \{0\}$. Note $B$ is uncountable by a slight variation of 4. above. Then $A - B = \{0\}$.

For (b), let $A = \mathbb{R}$ and $B = \mathbb{R} - \mathbb{Z}$. Note $B$ is uncountable by 4. Then $A - B = \mathbb{Z}$ which is countable.

For (c), let $A = \mathbb{R}$ and $B = \mathbb{R}_{<0}$. Note both sets are uncountable by Cantor diagonalization. Then $A - B = \mathbb{R}_{\geq 0}$ which is also uncountable.

**6.** Show that there is no surjective function $f$ from a set $S$ to its power set $\mathcal{P}(S)$. Hence conclude that $|S| < |\mathcal{P}(S)|$.

*Hint:* Suppose that $f : S \longrightarrow \mathcal{P}(S)$ is surjective. let $T = \{s \in S \mid s \notin f(s)\}$. Show that there is no element $s$ so that $f(s) = T$ (suppose there was, and do an argument reminiscent of the contradiction we get when we assert that "this statement is false").

**Solution:** Following the hint, suppose that $f(s) = T$ for some $s \in S$. Then there are two possibilities. Either $s \in T$ or $s \notin T$. If $s \in T$, then by the definition of $T$, $s \notin f(s) = T$, a contradiction. On the other hand if $s \notin T$, then $s \in f(s) = T$ by the definition of $T$, but that is also a contradiction.

**7.** Suppose that $\{A_1, A_2, A_3, \ldots\}$ is a countable list of countably infinite sets. Show that

$$A_1 \cup A_2 \cup A_3 \cup \cdots = \bigcup_{i \in \mathbb{Z}_{>0}} A_i$$

is also countable.

**Solution:** Write

$$\begin{aligned}
A_1 &= \{a_{1,1}, a_{1,2}, a_{1,3}, a_{1,4}, \ldots\} \\
A_2 &= \{a_{2,1}, a_{2,2}, a_{2,3}, a_{2,4}, \ldots\} \\
A_3 &= \{a_{3,1}, a_{3,2}, a_{3,3}, a_{3,4}, \ldots\} \\
A_4 &= \{a_{4,1}, a_{4,2}, a_{4,3}, a_{4,4}, \ldots\} \\
A_5 &= \ldots
\end{aligned}$$

and then do the same pattern we did when showing that $\mathbb{Q}_{>0}$ was countable.