WORKSHEET #1 – MATH 2200
SPRING 2018

SOLUTIONS

1. Show that if $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are functions such that $g \circ f$ is injective, then f is injective.

Solution: Assume that $g \circ f$ is injective. We want to show that f is injective, so suppose that $a, a' \in A$ are such that $f(a) = f(a')$ (we will be done if we can show that $a = a'$). Applying g to both sides of the equation we obtain that $g(f(a)) = g(f(a'))$. But $g(f(x)) = (g \circ f)(x)$ and so $(g \circ f)(a) = (g \circ f)(a')$. This implies that $a = a'$ using the assumption that $g \circ f$ is injective. But then we have just shown that f is injective too.

Remark: The thing I would have most liked to see on people's solutions is an explanation of how one goes from $f(a) = f(a')$ to $g(f(a)) = g(f(a'))$, in other words saying that they are applying g to both sides. This is **one of my favorite** problems to put on midterms and finals.

2. Give an example of functions $f : A \longrightarrow B$ and $g : B \longrightarrow C$ such that $g \circ f$ is injective but g is not injective.

Solution: There are lots of correct answers. You can set $A = \{1\}$, $B = \{2, 3\}$ and $C = \{4\}$. Then define $f : A \longrightarrow B$ by $f(1) = 2$ and $g : B \longrightarrow C$ by $g(2) = 4$ and $g(3) = 4$. Then g is not injective (since both $2, 3 \mapsto 4$) but $g \circ f$ is injective.

Here's another correct answer using more familiar functions.

Let $f : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$ be given by $f(x) = \sqrt{x}$. Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be given by $g(x) = x^2$. Then g is not injective (since $g(1) = g(-1)$) but $g \circ f : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$ is injective since it sends $x \mapsto x$.

Remark: Lots of groups did some variant of the second example. I took off points if they didn't specify the domain and codomain though. Note that the codomain of f **must equal** the domain of g for $g \circ f$ to make sense.

3. Suppose that $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are functions and that $g \circ f$ is surjective. Is it true that f must be surjective? Is it true that g must be surjective? Justify your answers with either a counterexample or a proof.

Solution: There are two questions in this problem.

Must f be surjective? The answer is no. Indeed, let $A = \{1\}$, $B = \{2, 3\}$ and $C = \{4\}$. Then define $f : A \longrightarrow B$ by $f(1) = 2$ and $g : B \longrightarrow C$ by $g(2) = 4$ and $g(3) = 4$. We see that $g \circ f : \{1\} \longrightarrow \{4\}$ is surjective (since $1 \mapsto 4$) but f is certainly not surjective.

Must g be surjective? The answer is yes, here's the proof. Suppose that $c \in C$ is arbitrary (we must find $b \in B$ so that $g(b) = c$, at which point we will be done). Since $g \circ f$ is surjective, for the c we have already fixed, there exists some $a \in A$ such that $c = (g \circ f)(a) = g(f(a))$. Let $b := f(a)$. Then $g(b) = g(f(a)) = c$ and we have found our desired b.

Remark: It is good to compare the answer to this problem to the answer to the two problems on the previous page.

The part of this problem most groups had the most issue with was the second. Everyone should be comfortable with carefully proving a function is surjective by the time we get to the midterm.

4. Suppose that $A = \{\star, 2, \emptyset\}$, $B = \{\{\emptyset\}, \{1, 2\}\}$. Let f be the function defined by $f(\star) = \{\emptyset\}$, $f(2) = \{\emptyset\}$ and $f(\emptyset) = \{1, 2\}$. What is the graph of f?

Solution: Remember, the graph of a function $f : A \longrightarrow B$ is the set of pairs $\{(x, f(x)) \mid x \in A\}$. Thus for us, the answer is

$$\{(\star, \{\emptyset\}), (2, \{\emptyset\}), (\emptyset, \{1, 2\})\}.$$

Remark: Many groups didn't seem to remember the definition of a graph.

5. Give an example of the following functions:

(a) A function $f : \mathbb{Z}_{>0} \longrightarrow \mathbb{Z}$ that is injective but not surjective.
(b) A function $g : \mathbb{Z}_{>0} \longrightarrow \mathbb{Z}$ that is surjective but not injective.
(c) A function $h : \mathbb{Z}_{>0} \longrightarrow \mathbb{Z}$ that is bijective.

Solution:

(a) You can let $f(x) = x$. It is not surjective since all the negative numbers are missed.

(b) This one is trickier. You need to hit all integers (both positive and *negative*). Most successful solutions to this problem are going to be multipart functions. Here's one that works (but it's definitely not the only solution).

$$g(x) = \begin{cases} x/2 & \text{if } x \text{ is even.} \\ (-x+3)/2 & \text{if } x \text{ is odd.} \end{cases}$$

First we should argue that this function is indeed surjective. For any integer $n > 0$, we notice that $g(2n) = n$. On the other hand if $n \leq 0$, then $-2n + 3 \geq 0$ and it is also odd. Hence $g(-2n + 3) = (-(-2n + 3) + 3)/2 = n$. Thus every integer is the g-image of something and so g is surjective.

To show it is not injective, simply notice that $g(2) = 1 = g(1)$.

(c) The solution to this one is similar to the solution to (b) (again, there are lots of other correct answers as well).

$$h(x) = \begin{cases} x/2 & \text{if } x \text{ is even.} \\ (-x+1)/2 & \text{if } x \text{ is odd.} \end{cases}$$

The explanation that h is surjective is essentially the same as for (b). To see it is injective, simply note that $x/2$ only outputs positive (> 0) integers while $(-x + 1)/2$ only outputs nonpositive (≤ 0) integers. So there can be no collisions.

Remark: The most common mistake was that people didn't notice that the codomain of these functions was supposed to be $\mathbb{Z}$. Thus $f(x) = \sqrt{x}$ won't work as an example for (a) since $\sqrt{3}$ is not in the codomain (it is not an integer).