WORKSHEET #3 – MATH 1260
FALL 2014

DUE WEDNESDAY, SEPTEMBER 17TH

For this assignment, you are allowed and encouraged to work in groups. Each group only has to turn in one assignment worksheet, but make sure it is done neatly.

Suppose we have a space curve parameterized by $t \mapsto \vec{r}(t) = \langle f(t), g(t), h(t)\rangle$. We saw in class on Wednesday that if we set $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$ then the derivative of $\vec{T}(t)$ was perpendicular to $\vec{T}(t)$. We set $\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$, this is called the *normal vector*.

1. Verify that $\vec{T}(t)$ and $\vec{T}'(t)$ are perpendicular again by writing down the argument here.

Solution: Since $|\vec{T}(t)| = 1$ we see that $\vec{T}(t)\cdot\vec{T}(t) = 1$. Thus differentiating both sides we get that $\vec{T}'(t)\cdot\vec{T}(t) + \vec{T}(t)\cdot\vec{T}'(t) = 0$ and hence that $2(\vec{T}(t)\cdot\vec{T}'(t)) = 0$ but then $\vec{T}(t)\cdot\vec{T}'(t) = 0$ which proves they are perpendicular as claimed.

The question that was brought up was, well then what direction is $\vec{T}'(t)$ pointing in?!? The goal of this worksheet is to provide an answer to that question.

2. Let's begin with an example. Consider the function $\vec{r}(t) = \langle \cos(t), \sin(t), t\rangle$. Compute $\vec{N}(t)$, including at some explicit examples, and then draw a picture showing the direction it is pointing. Do the same for the function $\vec{u}(t) = \langle t\cos(t), t\sin(t), t\rangle$.

Solution: I'm not going to draw any pictures but I can discuss the analysis. But, $\vec{r}'(t) = \langle -\sin(t), \cos(t), 1\rangle$ so that $|\vec{r}'(t)| = \sqrt{1+1} = \sqrt{2}$. Thus $\vec{T}(t) = \frac{1}{\sqrt{2}}\langle -\sin(t), \cos(t), 1\rangle$. Next $\vec{T}'(t) = \frac{1}{\sqrt{2}}\langle -\cos(t), -\sin(t), 0\rangle$ and $|\vec{T}'(t)| = \frac{1}{\sqrt{2}}$. Hence $\vec{N}(t) = \langle -\cos(t), -\sin(t), 0\rangle$. Let's analyze this. This is a vector pointing exactly inwards towards the z-axis (since it is exactly the x and y components of the negative of $\vec{r}(t)$).

Let's try the same computation with $\vec{u}(t)$. This becomes much much messier but I get

$$\vec{T}(t) = \left\langle \frac{(\cos(t) - t\sin(t))}{\sqrt{2+t^2}}, \frac{(t\cos(t) + \sin(t))}{\sqrt{2+t^2}}, \frac{1}{\sqrt{2+t^2}} \right\rangle$$

and unfortunately $\vec{N}(t)$ is really messy

$$\vec{N}(t) = \left\langle \frac{-t(3+t^2)\cos(t) - (4+t^2)\sin(t)}{\sqrt{(2+t^2)(8+5t^2+t^4)}}, \frac{(4+t^2)\cos(t) - t(3+t^2)\sin(t)}{\sqrt{(2+t^2)(8+5t^2+t^4)}}, \frac{-t}{\sqrt{(2+t^2)(8+5t^2+t^4)}} \right\rangle$$

This is hard to analyze, just looking at the z coordinate, it's always negative and for t very large the denominator is much bigger than the numerator. In other words, I am somehow curving downwards more for t small. In fact, for $t \gg 0$, the whole vector looks like

$$\vec{N}(t) \sim \langle -\cos(t), -\sin(t), 0\rangle$$

which is again just pointing inward at the z-axis. This makes sense. But on the other hand, let's imagine t is small (say less than $\frac{1}{1000}$). Let's assume any t^2 is about equal to zero, this simplifies things massively and we get.

$$\vec{N}(t) \sim \left\langle -t\frac{3}{4}\cos(t) - \sin(t), \cos(t) - t\frac{3}{4}\sin(t), \frac{-t}{4} \right\rangle.$$

If t is actually equal to zero this becomes $\langle 0, 1, 0\rangle$. This makes sense if you imagine how this curve actually turns into the origin. Ok, so analyzing the slightly less messy equation, we get that again the z coordinate shrinks as t goes to zero. You might then ask, when is the normal vector pointing the most downwards. This appears to occur at around $t = 1.13669$.

3. Suppose that $\vec{r}(t)$ is the equation of a line. What is $\vec{T}'(t)$, does $\vec{N}(t)$ even make sense?

Hint: You can try an example like $\vec{r}(t) = \langle 1, 2, 3\rangle t + \langle 4, 5, 6\rangle$ if the answer isn't clear to you.

Solution: Then obviously $\vec{T}'(t)$ is the unit vector pointing in the direction of the line. It is *constant* as t varies and so $\vec{T}'(t) = 0$. Hence $\vec{N}(t)$ doesn't even make sense (division by zero). This is *OK*, the line isn't curving at all.

4. Imagine that $\vec{r}(t)$ is the position of a particle at time t. Then $\vec{r}'(t) = \vec{v}(t)$ is the velocity vector of the particle and $\vec{r}''(t) = \vec{a}(t)$ is the acceleration vector. Show that $\vec{a}(t)$ is in the span of $\vec{T}(t)$ and $\vec{N}(t)$.

Hint: Let $v(t) = |\vec{v}(t)|$ be the speed function. Consider then the formula $v(t)\vec{T}(t) = \vec{v}(t)$ and differentiate it.

Solution: Following the hint we get that $v'(t)\vec{T}(t) + v(t)\vec{T}'(t) = \vec{v}'(t) = \vec{a}(t)$. But $\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$ and so

$$\vec{a}(t) = v'(t)\vec{T}(t) + (v(t)\ |\vec{T}'(t)|)\vec{N}(t)$$

which that $\vec{a}$ is in the span of $\vec{T}(t)$ and $\vec{N}(t)$

5. Explain in words why this must mean that $\vec{T}'(t)$ is pointing along the radial line of the acceleration.

Solution: Well, $\vec{N}(t)$ is perpendicular to $\vec{T}(t)$ and so since $\vec{a}(t)$ can be written in terms of the change in speed times the velocity direction plus a positive constant times $\vec{N}(t)$. This implies that $\vec{N}(t)$ must point in a direction that you are accelerating.

6. Consider now the formula for curvature $\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$. Use this and the work you did in **4.** to derive the formula

$$\vec{a}(t) = v'(t)\vec{T}(t) + \kappa(t)(v(t))^2\vec{N}(t).$$

Solution: We are already really close. Notice that $|\vec{T}'(t)| = \kappa(t)(|\vec{r}'(t)|) = \kappa(t)v(t)$. Plugging this into our solution for **4.** we get

$$\vec{a}(t) = v'(t)\vec{T}(t) + (v(t)\ |\vec{T}'(t)|)\vec{N}(t) = v'(t)\vec{T}(t) + (v(t)\ |\vec{T}'(t)|)\vec{N}(t) = v'(t)\vec{T}(t) + \kappa(t)(v(t))^2\vec{N}(t)$$

which is exactly what we wanted.