WORKSHEET #1 – MATH 1260
FALL 2014

DUE TUESDAY, SEPTEMBER 2ND

For this assignment, you are allowed and encouraged to work in groups. Each group only has to turn in one assignment worksheet, but make sure it is done neatly.

We'll learn some things about vectors that are not in the book.

Suppose $\vec{v}_1, \ldots, \vec{v}_m$ are in $\mathbb{R}^n$. We say that these vectors *span* $\mathbb{R}^n$ if every vector $\vec{w}$ in $\mathbb{R}^n$ can be written as

$$\vec{w} = a_1\vec{v}_1 + \ldots + a_m\vec{v}_m.$$

for some real numbers $a_i \in \mathbb{R}$ (such a sum with real coefficients is called a *linear combination*). Geometrically, this means that the vectors don't all lie in the same hyperplane. (So in $\mathbb{R}^3$, it means they don't all lie in a plane).

1. Show that $\vec{i} = \langle 1, 0\rangle, \vec{j} = \langle 0, 1\rangle$ span $\mathbb{R}^2$.

Hint: Given $\vec{u} = \langle a, b\rangle$, how do you write it as a linear combination of $\vec{i}$ and $\vec{j}$?

Solution: We can write $\vec{u} = a\vec{i} + b\vec{j} = a\langle 1, 0\rangle + \langle 0, 1\rangle$. That's it.

2. Do $\vec{i}, \vec{j},$ and $\vec{w} = \langle 3, 2, -1\rangle$ span $\mathbb{R}^3$? If so, prove it (give a justification).

Solution: Yes they span $\mathbb{R}^3$. Indeed, given $\vec{u} = \langle a, b, c\rangle$ we can write

$$(a + 3c)\vec{i} + (b + 2c)\vec{j} + (-c)\vec{w} = \langle a + 3c - 3c, b + 2c - 2c, c\rangle = \vec{u}$$

as desired.

3. Show that $\vec{u} = \langle 1, 0, 1\rangle, \vec{v} = \langle 2, 1, 0\rangle,$ and $\vec{w} = \langle -1, -1, 1\rangle$ do not span $\mathbb{R}^3$.

Hint: Two vectors that are not collinear obviously span a plane. Find a vector that cannot be written as a linear combination of the three vectors above.

Solution: Consider the vector $\vec{k} = \langle 0, 0, 1\rangle$. Suppose that $a\vec{u} + b\vec{v} + c\vec{w} = \vec{k}$. Then

$$\langle a + 2b - c, 0 \cdot a + b - c, a + 0 \cdot b + c\rangle = \vec{k}$$

and so

$$\begin{aligned} a + 2b - c &= 1 \\ b - c &= 0 \\ a + c &= 0 \end{aligned}$$

and the bottom two equations implies that $b = c$ and $a = -c$. Plugging this into the first equation gives $-c + 2c - c = 1$, but the left side is zero and so $0 = 1$. This is a contradiction and so $\vec{u}, \vec{v}, \vec{w}$ cannot span $\mathbb{R}^3$.

4. Show that $\vec{i}, \vec{j}, \vec{k},$ and $\vec{w} = \langle 1, 2, 3\rangle$ span $\mathbb{R}^3$.

Solution: Obviously $\vec{i}, \vec{j}, \vec{k}$ span $\mathbb{R}^3$ using the same idea as in problem **1.**. Adding $\vec{w}$ doesn't stop it from spanning, indeed we can always write

$$\langle a, b, c\rangle = a\vec{i} + b\vec{j} + c\vec{k} + 0\vec{w}.$$

Another way to say that $\vec{v}_1, \ldots, \vec{v}_m$ span $\mathbb{R}^n$ is to say that every other vector $\vec{u} \in \mathbb{R}^n$ can be written as a linear combination of the $\vec{v}_i$ in *at least one way.* This leads us to our next notion.

We say that $\vec{v}_1, \ldots, \vec{v}_m$ are *linearly independent* in $\mathbb{R}^n$ if for each vector $\vec{u} \in \mathbb{R}^n$, there exists *at most one* linear combination of the $\vec{v}_i$ that equals $\vec{u}$. In other words, if there are at most one set of real numbers $a_1, \ldots, a_m \in \mathbb{R}^n$ so that

$$\vec{u} = a_1\vec{v}_1 + \ldots + a_m\vec{v}_m.$$

By the way, sets of vectors that are not linear independent are called *linearly dependent.*

5. Show that $\vec{i}, \vec{j}, \vec{k}$ form a linearly independent set of vectors in $\mathbb{R}^3$.

Hint: Suppose that we can write

$$a_1\vec{i} + b_1\vec{j} + c_1\vec{k} = a_2\vec{i} + b_2\vec{j} + c_2\vec{k}.$$

Then write each side as a single vector $\langle \ldots, \ldots, \ldots, \rangle$ and deduce that $a_1 = a_2, b_1 = b_2, c_1 = c_2$. Why is doing this enough?

Solution: If $a_1\vec{i} + b_1\vec{j} + c_1\vec{k} = a_2\vec{i} + b_2\vec{j} + c_2\vec{k}$ then $\langle a_1, b_1, c_1\rangle = \langle a_2, b_2, c_2\rangle$ and hence $a_1 = a_2, b_1 = b_2, c_1 = c_2$. This shows that there is exactly one way to write any given vector as a linear combination of $\vec{i}, \vec{j}, \vec{k}$ which proves the result.

6. Find two linearly dependent vectors in $\mathbb{R}^2$. Do they span $\mathbb{R}^2$?

Solution: Consider $\vec{u} = \langle 1, 1\rangle$ and $\vec{v} = \langle 2, 2\rangle$. They are certainly linearly dependent since $\vec{v} = 2\vec{u}$. Alternately, we can write the vector $\vec{0}$ in two ways, as $0\vec{u} + 0\vec{v}$ or as $2\vec{u} + (-1)\vec{v}$.

They do not span $\mathbb{R}^2$ since they are collinear and hence every linear combination of them also lies on the same line.

Alternately, they do not span $\mathbb{R}^2$ since $\vec{i} = \langle 1, 0\rangle$ is not a linear combination of them, indeed if $\vec{i} = a\vec{u} + b\vec{v} = (a + 2b)\vec{u}$ then $\vec{i}$ is parallel to $\vec{u}$ which it obviously is not.

7. Find three linearly dependent vectors in $\mathbb{R}^2$ that span $\mathbb{R}^2$.

Solution: Consider $\vec{i}, \vec{j}, \vec{u} = \langle 1, 1\rangle$. Obviously $\vec{i}$ and $\vec{j}$ already span $\mathbb{R}^2$ and so using the same argument as in **4.** so do $\vec{i}, \vec{j}, \vec{u}$. On the other hand $\vec{u}$ is a linear combination of $\vec{i}$ and $\vec{j}$ and so they form a linearly dependent set.

8. Find three different linearly dependent vectors in $\mathbb{R}^3$.

Solution: Consider $\vec{i}, \vec{j}, \vec{u} = \langle 1, 1, 0\rangle$. They are linearly dependent since $\vec{u} = \vec{i} + \vec{j}$.

9. Are the vectors $\vec{x} = \langle 1, 0, 0, 1\rangle, \vec{y} = \langle 0, 1, 1, 0\rangle, \vec{z} = \langle 0, 0, 2, 1\rangle, \vec{w} = \langle 0, 0, 0, -1\rangle$ linearly independent in $\mathbb{R}^4$? Do they span $\mathbb{R}^4$?

Solution: They are linearly independent. To check this suppose that first $a_1\vec{x}+b_1\vec{y}+c_1\vec{z}+d_1\vec{w} = a_2\vec{x} + b_2\vec{y} + c_2\vec{z} + d_2\vec{w}$ and so setting $a = a_1 - a_2, b = b_1 - b_2$ etc. we have that

$$a\vec{x} + b\vec{y} + c\vec{z} + d\vec{w} = \vec{0}.$$

Writing this out into a system of equations yields

$$\begin{aligned} a &= 0 \\ b &= 0 \\ b + 2c &= 0 \\ a + c - d &= 0 \end{aligned}$$

Since $a, b = 0$, the third equation implies that $c = 0$. Then also the final equation implies that $d = 0$. Hence $a_1 - a_2 = 0, b_1 - b_2 = 0, \ldots$ and so $a_1 = a_2, b_1 = b_2, c_1 = c_2, d_1 = d_2$ which proves that the vectors are linearly independent.

To show they span $\mathbb{R}^4$ consider the equation

$$a\vec{x} + b\vec{y} + c\vec{z} + d\vec{w} = \langle m, n, o, p\rangle.$$

We want to solve for a, b, c, d which turns into a system of equations

$$\begin{aligned} a &= m \\ b &= n \\ b + 2c &= o \\ a + c - d &= p \end{aligned}$$

Plugging the second equation into the third tells us that $c = \frac{o-n}{2}$. Plugging our values for a, c, d into the final equation yields $d = m + \frac{o-n}{2} - p$. Hence we have the solutions

$$\begin{aligned} a &= m \\ b &= n \\ b &= \tfrac{o-n}{2} \\ d &= m + \tfrac{o-n}{2} - p \end{aligned}$$

This shows that every vector $\langle m, n, o, p\rangle$ is a linear combination of $\vec{x}, \vec{y}, \vec{z}, \vec{w}$ and so they form a spanning set as claimed.

A linearly independent spanning set is called a *basis*. Here are some facts about bases. You may take these as given going forward.

(1) Any basis in $\mathbb{R}^n$ has exactly n elements.
(2) Any linearly independent set of n vectors in $\mathbb{R}^n$ is automatically a spanning set (and hence a basis).
(3) Any spanning set of n vectors in $\mathbb{R}^n$ is automatically linearly independent (and hence a basis).