WORKSHEET #13 – MATH 1260
FALL 2014

NOT DUE

1. Short answer:

(a) Find the equation of the tangent plane to $z = x^2 + y^2$ at the point $\langle 1, 1, 2\rangle$.

Solution: $z_x(1,1) = 2x = 2$, $z_y(1,1) = 2y = 2$. So then the tangent plane equation is $2(x-1) + 2(y-1) + 2 = z$.

(b) Find the equation of the tangent line to $\vec{r}(t) = \langle t, t^2, t+1\rangle$ at the point $\langle 1, 1, 2\rangle$.

Solution: The point $\langle 1, 1, 2\rangle$ corresponds to $t = 1$. Now, $\vec{r}'(1) = \langle 1, 2, 1\rangle$. So our tangent line equation is $\langle 1, 2, 1\rangle t + \langle 1, 1, 2\rangle$.

(c) Compute the curvature of $\vec{r}(t) = \langle t, t^2, t+1\rangle$ at $\langle 1, 1, 2\rangle$.

Solution: We already computed $\vec{r}'(1) = \langle 1, 2, 1\rangle$, let's compute $\vec{r}''(1) = \langle 0, 2, 0\rangle$. Then

$$\kappa(1) = \frac{|\vec{r}'(1) \times \vec{r}''(1)|}{|\vec{r}'(1)|^3} = \frac{|\langle -2, 0, 2\rangle|}{6^{3/2}} = \frac{8^{1/2}}{6^{3/2}} = \frac{1}{3^{3/2}}$$

(d) What is the distance between the plane $x + 3y - 4z = 2$ and the point $\langle 1, 1, 1\rangle$.

Solution: The normal to the plane is $\langle 1, 3, -4\rangle$. So we form the line $\langle 1, 3, -4\rangle t + \langle 1, 1, 1\rangle$. This can be written as $x = t+1, y = 3t+1, z = -4t+1$. Plugging these into our plane equation gives us:

$$(t+1) + 3(3t+1) - 4(-4t+1) = 2 \text{ or } 26t = 2 \text{ or } t = 1/13.$$

Plugging this back into our equation gives us the intersection point $x = 14/13, y = 16/13, z = 9/13$. The distance between this point and $\langle 1, 1, 1\rangle$ is $\sqrt{(1/13)^2 + (3/13)^2 + (-4/13)^2}$.

(e) Suppose the temperature of a hot plate at position $\langle x, y\rangle$ is given by $z = ye^{xy} + y^2 + 2x + 120$. If an intrepid ant is at the origin, what direction should the ant move in order to lower the temperature on his feet?

Solution: The gradient is $\langle y^2 e^{xy} + 2, e^{xy} + xye^{xy} + 2y\rangle$. At the origin $x = 0, y = 0$, this becomes $\langle 2, 0\rangle$. Hence the ant should move in the direction opposite this, or $\langle -1, 0\rangle$.

(f) Give an example of 3 vectors in $\mathbb{R}^3$ that do not form a basis for $\mathbb{R}^3$.

Solution: $\langle 1, 1, 1\rangle, \langle 2, 2, 2\rangle$ and $\langle 3, 3, 3\rangle$ would work since the three vectors are collinear and thus not linearly independent, and so they can't form a basis.

2. Short answer:

(a) Give an example of a function $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that is not one-to-one.

Solution: $\langle x, y\rangle \mapsto \langle 1, 1\rangle$ it is not one-to-one since $\langle 0, 0\rangle$ and $\langle 0, 1\rangle$ both get sent to the same output.

(b) Setup an integral that would compute the arclength of the curve $\vec{r}(t) = \langle t^2, t^3, t^4\rangle$ between $t = 1$ and $t = 3$.

Solution: First $\vec{r}'(t) = \langle 2t, 3t^2, 4t^3\rangle$ and so the arclength is

$$\int_1^3 \sqrt{(2t)^2 + (3t^2)^2 + (4t^3)^2}dt$$

(c) Setup an integral that would compute the mass of the region below the curve $z = (1-x-y)^3$, above the $z = 0$ plane and with $x, y \geq 0$.

Solution: Note that $(1 - x - y)^3 \geq 0$ when $1 - x - y \geq 0$. Hence we have

$$\int_0^1 \int_0^{1-x} \int_0^{(1-x-y)^3} 1dzdydx.$$

(d) Give an example of a vector field on $\mathbb{R}^3$ that is not the gradient of a function $w = f(x, y, z)$.

Solution: Consider $\langle y, 0, 0\rangle$. The curl is $\langle 0, 0, -1\rangle$ which is not zero. So this vector field can't be a gradient.

(e) Is the following vector field the curl of another vector field? $\vec{F} = x^2\vec{i} + (z^3 - 2yx)\vec{j} + x\vec{k}$.

Solution: $\text{div}\vec{F} = 2x - 2x + 0 = 0$ so the answer is yes.

(f) Consider the surface S parameterized by $r(u, v) = \langle u^2, e^{uv}, v^2\rangle$. Setup an integral to compute the surface area of S over the region where $0 \leq u \leq 1$ and $-1 \leq v \leq 1$.

Solution: $\vec{r}_u = \langle 2u, e^{uv}, 0\rangle, \vec{r}_v = \langle 0, e^{uv}, 2v\rangle$. Then the surface area is

$$\int_{-1}^1 \int_0^1 |\langle 2u, e^{uv}, 0\rangle \times \langle 0, e^{uv}, 2v\rangle|dudv = \int_{-1}^1 \int_0^1 |\langle 2ve^{uv}, 4uv, 2ue^{uv}\rangle|dudv$$

(g) Suppose that $\vec{F}$ is a conservative vector field on $\mathbb{R}^2$. Let C_1 be the curve parameterized by $\vec{u}(t) = \langle t^2, 2t\rangle$ for $0 \leq t \leq 1$ and let C_2 be the curve parameterized by $\vec{v}(t) = \langle \sin((\pi/4)t), 3^{t/2} - 1\rangle$ for $0 \leq t \leq 2$. Is

$$\int_{C_1} \vec{F} \cdot ds = \int_{C_2} \vec{F} \cdot ds?$$

Solution: There wasn't enough information in the first statement, this is fixed. Note $\vec{u}(0) = \langle 0, 0\rangle, \vec{u}(1) = \langle 1, 2\rangle$ and $\vec{v}(0) = \langle 0, 0\rangle, \vec{v}(1) = \langle 1, 2\rangle$. Thus the two curves have the same end points, and so we that the two integrals are equal by independence of path (since the vector field is conservative).

3. Short answer:

(a) Give an intuitive description of what the inverse function theorem says. Include pictures.

Solution: I'm not going to include pictures, but the inverse function theorem says that roughly that if the determinant of the Jacobian of a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is nonzero at a point $Q \in \mathbb{R}^n$ (and the partials of f are continuous), then f is invertible in at least near Q.

(b) Precisely state the implicit function theorem.

Solution: Suppose that $\mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$,

$$f(x_1, \ldots, x_n, y_1, \ldots, y_m) = \langle g_1(x_1, \ldots, y_m), \ldots, g_m(x_1, \ldots, y_m)$$

is a function with continuous partial derivatives. Consider the matrix

$$\begin{bmatrix} \frac{\partial g_1}{\partial y_1}, & \ldots, & \frac{g_m}{\partial y_1} \\ \ldots, & \ldots, & \ldots \\ \frac{\partial g_1}{\partial y_m}, & \ldots, & \frac{g_m}{\partial y_m} \end{bmatrix}$$

If the determinant of this matrix is nonzero at $Q = \langle \vec{a}, \vec{b} \rangle \in \mathbb{R}^n \times \mathbb{R}^m$, then there is an open set $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$, and there is a function $g : A \longrightarrow B$ such that for each $\vec{x} \in A$ there is a unique $\vec{y} \in B$ such that $g(\vec{x}, \vec{y}) = 0$.

(c) Give an example of a subset of $\mathbb{R}^2$ that is neither open nor closed.

Solution: For instance, consider $U = (0, 1] \times (0, 1) = \{(x, y) \mid 0 < x \leq 1, 0 < y < 1\}$.

(d) Is the set of integers (whole numbers) a compact subset of $\mathbb{R}$?

Solution: No, the integers are definitely not bounded, and compact sets are bounded.

(e) What does Green's theorem say?

Solution: Suppose that C is a piece-wise smooth closed curve, oriented counter-clockwise, bounding a region $D \subseteq \mathbb{R}^2$. If $\vec{F} = \langle P, Q \rangle$ is a vector field with continuous partials, then

$$\int_C \vec{F} \cdot ds = \iint_D (Q_x - P_y) dA.$$

(f) Give an example of a function $\mathbb{R}^3 \longrightarrow \mathbb{R}$ that is onto.

Solution: Consider the function $\langle x, y, z \rangle \mapsto x$. This is onto because for each real number x, we know that $\langle x, 0, 0 \rangle$ gets sent to x (in other words, hits x).

(g) Is the following parameterization of a curve a parameterization with respect to arclength?

$$\vec{r}(t) = \frac{1}{\sqrt{2}} \langle \cos(t), \sin(t), t \rangle$$

Solution: We compute $\vec{r}'(t) = \frac{1}{\sqrt{2}} \langle -\sin(t), \cos(t), 1 \rangle$. We observe that

$$|\vec{r}'(t)| = \sqrt{\frac{1}{2}(\sin(t))^2 + \frac{1}{2}(\cos(t))^2 + \frac{1}{2}(1)^2} = 1.$$

Since $|\vec{r}'(t)|$ is the constant 1, this curve is indeed parameterized via arc length.

4. Find the distance between the plane $x + 2y + 3z = 0$ and the point $\langle 4, 3, 2\rangle$.

Solution: The normal vector is $\langle 1, 2, 3\rangle$. We consider the line $\langle 1, 2, 3\rangle t + \langle 4, 3, 2\rangle$, which is $x = t + 4, y = 2t + 3, z = 3t + 2$. Plugging this into the equation yields

$$(t + 4) + 2(2t + 3) + 3(3t + 2) = 0 \text{ or } 14t = -16 \text{ or } t = -8/7.$$

Thus the point on the plane and on this line is $x = 20/7, y = 5/7, z = -10/7$. We find the distance between $\langle 4, 3, 2\rangle$ and this point. That is

$$|\langle 4, 3, 2\rangle - \langle 20/7, 5/7, -10/7\rangle| = |\langle 8/7, 16/7, 24/7\rangle| = \frac{1}{7}\sqrt{64 + 256 + 576} = \frac{1}{7}\sqrt{896} = \sqrt{128/7}.$$

(you don't have to simplify it this far).

5. Consider the vector field $\vec{F} = \langle -x, y\rangle$. Find a line L and a parametric representation $\vec{r}(t)$ of it so that the tangent vector $\vec{r}'$ of the line at any point $Q = \langle a, b\rangle$ has a multiple equal to $\vec{F}(\langle a, b\rangle)$. Bonus points if you can parameterize L in a non-linear way so that $\vec{r}'(Q) = \vec{F}(Q)$.

Solution: (The original version had a typo).

There are a couple solutions here. One obvious solution is the line $\vec{r}(t) = \langle 0, t\rangle$. Notice that $\vec{r}'(t) = \langle 0, 1\rangle$ which points in the same direction as $\vec{F}(0, y) = \langle 0, t\rangle$. To make $\vec{r}'(t) = \langle 0, y\rangle$, we need a function $y = g(t)$ with $g'(g^{-1}(y)) = y$. Here's one function that will work for this (at least for the top half of the line). Let $g(t) = e^t$, so $g^{-1}(y) = \ln(y)$. This only of course parameterizes the line for the positive part.

6. Find the distance of the point $(1, 2, 3)$ from the tangent plane to the surface parameterized by $\vec{r}(u, v) = \langle u^2, uv, v^3\rangle$ at $(1, 11)$.

Solution: The point $(1, 1, 1)$ corresponds to $v = 1$ and $u = 1$. So we compute $\vec{r}_u = \langle 2u, v, 0\rangle$ and $\vec{r}_v = \langle 0, u, 3v^2\rangle$.. At $\langle 1, 1\rangle$ this makes $\vec{r}_u(1, 1) = \langle 2, 1, 0\rangle$ and $\vec{r}_v(1, 1) = \langle 0, 1, 3\rangle$. The cross product of these two vectors is normal to the tangent plane:

$$\langle 2, 1, 0\rangle \times \langle 0, 1, 3\rangle = \langle 3, -6, 2\rangle.$$

So the plane equation is $3x - 6y + 2z = -1$ (plug in $\langle 1, 1, 1\rangle$). We consider the line $\langle 3, -6, 2\rangle t + \langle 1, 2, 3\rangle$, or $x = 3t + 1, y = -6t + 2, z = 2t + 3$. Plugging this into our plane equation we get $3(3t + 1) - 6(-6t + 2) + 2(2t + 3) = -1$ so that $49t - 3 = -1$ then $t = 2/49$. Thus $x = 55/49, y = 86/49, z = 151/49$. We subtract this from $\langle 1, 2, 3\rangle$ and we get

$$|\langle 55/49, 86/49, 151/49\rangle - \langle 1, 2, 3\rangle| = |\langle 6/49, -12/49, 2/49\rangle| = \frac{1}{49}\sqrt{36 + 144 + 4} = \frac{1}{49}\sqrt{184}.$$

Thus $\frac{1}{49}\sqrt{184}$ is the distance to the plane.

7. Recall that the transformation for rotation by θ degrees is given by $\vec{r}(u,v) = \langle u\cos(\theta) + v\sin(\theta), -u\sin(\theta) + v\cos(\theta)\rangle$. Use this to deduce the formula for $\sin(\alpha+\beta)$.

Solution: Consider $\vec{r}_\alpha$ and $\vec{r}_\beta$ defined as above. We know $\vec{r}_\alpha \circ \vec{r}_\beta = \vec{r}_{\alpha+\beta}$ since rotation by β followed by rotation by α is rotation by $\alpha+\beta$. On the other hand

$$\begin{aligned}
& u\cos(\alpha+\beta) + v\sin(\alpha+\beta), -u\sin(\alpha+\beta) + v\cos(\alpha+\beta) \\
= \; & \vec{r}_{\alpha+\beta}(u,v) \\
= \; & \vec{r}_\alpha \circ \vec{r}_\beta(u,v) \\
= \; & \vec{r}_\alpha(\langle u\cos(\beta) + v\sin(\beta), -u\sin(\beta) + v\cos(\beta)\rangle) \\
= \; & \langle (u\cos(\beta) + v\sin(\beta))\cos(\alpha) + (-u\sin(\beta) + v\cos(\beta))\sin(\alpha), \\
& -(\langle u\cos(\beta) + v\sin(\beta))\sin(\alpha) + (-u\sin(\beta) + v\cos(\beta))\cos(\alpha)\rangle
\end{aligned}$$

Lets plug in $u = 0, v = 1$. Then from the first coordinate we have

$$\sin(\alpha+\beta) = \sin(\beta)\cos(\alpha) + \cos(\beta)\sin(\alpha).$$

8. The base of an aquarium of volume V is made of stone and the sides are glass (there is no top). If stone costs 8 times as much as glass, what dimensions should the aquarium be (in terms of V) in order to minimze the cost of materials. Justify your answer. (You can use whatever method you want).

Solution: We let x, y, z denote the lengths of sides. We note that $V = xyz$ so $z = V/(xy)$ and the cost is $C(x, yz,) = 8xy + 2xz + 2yz = 8xy + 2V/y + 2V/x$. Let's look for critical points: $C_x = 8y - 2V/x^2, C_y = 8x - 2V/y^2$. We set these equal to zero and get $2V = 8yx^2, 2V = 8xy^2$. Thus $8yx^2 = 8xy^2$. Since we want $x, y > 0$, we obtain $x = y$. So then $V = 4x^3$. Solving for x we get $x = y = (V/4)^{1/3}$ and so $z = V/((V/4)^{2/3})$. Finally, we argue that this really is a minimum cost. $C_{xx} = 4V/x^3, C_{yy} = 4V/y^3, C_{xy} = 8$. Note then that $D = C_{xx}C_{yy} - C_{xy}^2 = 16V^2/(x^3y^3) - 64$. We plug in our x, y values and get $D = 16V^2/((V/4)^2) - 64 = 256 - 64 = 192 > 0$. On the other hand, $C_{xx} > 0$ so this really is a min.

You could also do the Lagrange multiplier method.

9. Reparameterize the curve $t \mapsto \langle \cos(2\pi t), \sin(2\pi t), t\rangle$ with respect to arc length.

Solution: Let's first compute

$$|\vec{r}'(t)| = |\langle -2\pi\sin(2\pi t), 2\pi\cos(2\pi t), 1\rangle| = \sqrt{(2\pi)^2 + 1}.$$

This is a constant, so reparameterization with respect to arclength is easy. Indeed set $\vec{u}(s) = \vec{r}(\frac{s}{\sqrt{(2\pi)^2+1}})$. Then $|\vec{u}'(s)| = 1$ and so we have parameterized with respect to arc length.

10. Consider the integral

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} dzdydx$$

it gives the volume of some region. Draw the region, and then setup the same integral both polar and in spherical coordinates. (You don't need to evaluate the integral, unless you want to).

Solution: I won't draw it here, but I will describe it. The top z-term obviously defines a sphere of radius $\sqrt{2}$. The bottom z-term describes a cone. Setting these two terms equal quickly yields $2 = 2x^2 + 2y^2$ or $1 = x^2 + y^2$. Thus the top of the sphere and the cone intersect in a circle of radius 1 in the $z = 1$ plane. We have made an *ice-cream-cone*!

Ok, in cylindrical coordinates $x = r\cos\theta, y = r\sin\theta$ this becomes

$$\int_{0}^{2\pi}\int_{0}^{1}\int_{r}^{\sqrt{2-r^2}} rdzdrd\theta.$$

In spherical coordinates $x = \rho\cos(\theta)\sin(\phi), y = \rho\sin(\theta)\sin(\phi), z = \rho\cos(\phi)$, this becomes

$$\int_{0}^{2\pi}\int_{0}^{\pi/4}\int_{0}^{\sqrt{2}} \rho^2\sin(\phi)d\rho d\phi d\theta.$$

Note the Jacobian term in each of the above.

11. Consider the region above the plane $z = -2$, below the plane $z = 4y$ and inside the cylinder $x^2 + y^2 = 1$ (in other words, $z \geq -2, z \leq 4y, x^2 + y^1 \leq 1$). Draw the region. Suppose now that the density of the object is given by the formula $\rho(x, y, z) = 5x^2e^yz$. Setup, but do not evaluate an integral that computes the mass of the region.

Solution: Ok, so this will be yet another doorstop. Note the two planes intersect at $y = -\frac{1}{2}$. Thus we have the following integral.

$$\int_{-\frac{1}{2}}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{-2}^{4y} 5x^2e^yzdzdxdy$$

12. Consider a particle moving along a curve from $(0,1)$ to $(1,0)$ via the parameterization $\vec{r}(t) = \langle t^2, e^{t(1-t)}(1-t)^{49}\rangle$ for $t = 0$ to 1. Find the work done by the force field $\vec{F} = y^2\vec{i} + 2xy\vec{j}$ as the particle moves along this curve segment.

Solution: First notice that $\vec{F} = P\vec{i} + Q\vec{j}$ satisfies $Q_x - P_y = 2y - 2y = 0$ so we see that $\vec{F}$ is conservative. The most obvious thing to then do is find out what it is the gradient of. Note that if $f(x,y) = xy^2$, then $\nabla f = \vec{F}$. So by the fundamental theorem of line integrals, we know that

$$\int_C \vec{F} \cdot ds = f(\vec{r}(1)) - f(\vec{r}(0)) = f(\langle 1,0\rangle) - f(\langle 0,1\rangle) = 0 - 0 = 0.$$

Another reasonable option would be to find another curve with the same start and end points and compute an integral over that.

13. Consider the wire parameterized by the formula $\vec{r}(t) = \langle t\cos(t), \sin(t)\rangle$, for $t = 0$ to $t = 2\pi$. Suppose the mass of the wire per unit length at point (x,y) is given by $\mu(x,y) = x^2 + y^2 + 1$. Setup an integral to compute the mass of the wire.

Solution: Note

$$\begin{aligned} & |\vec{r}'(t)| \\ = \; & |\langle \cos(t) - t\sin(t), \cos(t)\rangle| \\ = \; & \sqrt{(\cos(t))^2 - 2t\cos(t)\sin(t) + t^2(\sin(t))^2 + (\cos(t))^2} \\ = \; & \sqrt{2(\cos(t))^2 - 2t\cos(t)\sin(t) + t^2(\sin(t))^2} \end{aligned}$$

Additionally observe that $x^2 + y^2 + 1 = (t\cos(t))^2 + (\sin(t))^2 + 1$. Putting this together gives us our integral.

$$\int_0^{2\pi} ((t\cos(t))^2 + (\sin(t))^2 + 1)\sqrt{2(\cos(t))^2 - 2t\cos(t)\sin(t) + t^2(\sin(t))^2}dt$$

14. Let S be the surface defined by the equation

$$z = x(1-x)y(1-y)(7+\sin(xy)+e^{\cos(x)})$$

and above the square $0 \le x \le 1$, $0 \le y \le 1$. Let $\vec{F}$ be the vector field $y^2\vec{i} - \vec{j} + x\vec{k}$. Compute the flux integral

$$\iint_S \vec{F} \cdot dS.$$

Solution: We notice first that $\div(\vec{F}) = 0$ and so our integral is independent of surface (at least over surfaces with the same boundary). Fortunately, our surface has boundary equal to the square T with corners $\langle 0,0,0\rangle, \langle 0,1,0\rangle, \langle 1,0,0\rangle, \langle 1,1,0\rangle$. Hence our integral becomes

$$\iint_T \vec{F} \cdot dS.$$

If we parameterize our surface by $\langle u,v\rangle \mapsto \langle u,v,0\rangle$ then the normal to the surface (the cross product of the partials of our parameterization) is easily seen to be the constant $\langle 0,0,1\rangle$. When we dot this with $\vec{F}$ we get $x = u$. Hence we integrate and

$$\int_0^1 \int_0^1 ududv = \int_0^1 (\frac{1}{2}u^2)\Big|_0^1 dv = \int_0^1 \frac{1}{2}dv = \frac{1}{2}$$

is our answer.

15. Let S_1 be the surface defined by $z = 0, x \ge 0, y \ge 0$, and $x + y \le 1$. Let S_2 be the surface defined by $z = e^{\cos(x)}x^4y^2(1-x-y)^3, z \ge 0, x \ge 0, y \ge 0$. Suppose that the volume of the region below S_2 and above S_1 is K. If $\vec{F} = \langle \frac{1}{K}x, z, 2y\rangle$, compute

$$\iint_{S_2} \vec{F} \cdot dS.$$

Solution: Ok, so S_1 is a triangle on the xy-plane, S_2 is uglier, but fortunately the z-value for S_2 is zero along the boundary of S_1 and so we see that that S_2 and S_1 have the same boundary.

First we compute the integral over S_1, noting that if we parameterize it by $\langle u,v\rangle \mapsto \langle u,v,0\rangle$ then the normal vector is again $\langle 0,0,1\rangle$ just as before. Thus

$$\iint_{S_1} \vec{F} \cdot dS = \iint_{S_1} 2ydudv = \int_0^1 \int_0^{1-u} 2vdvdu = \int_0^1 v^2\Big|_0^{1-u} du = \int_0^1 ((1-u)^2 - 0)du = (\frac{1}{3}u^3 - u^2 + u)\Big|_0^1 = \frac{1}{3}.$$

Unfortunately, the integrals over S_1 and S_2 are not equal since the divergence of $\vec{F}$ is not zero. However, we have $\div(\vec{F}) = \frac{1}{K}$ and so if we let E be the region below S_2 and above S_1 then

$$\iint_{S_2} \vec{F} \cdot dS - \iint_{S_1} \vec{F} \cdot dS = \iiint_E \div(\vec{F})dV = \iiint_E \frac{1}{K}dV = K\frac{1}{K} = 1.$$

It follows that $\iint_{S_2} \vec{F} \cdot dS = 1 + \frac{1}{3} = \frac{4}{3}$.

16. State and prove the Heine-Borel theorem. You may (and should) use the following fact.

Fact: If $B \subseteq \mathbb{R}$ is a *nonempty* bounded set, then B has a least upper bound. In other words there is a number $L > 0$ so that $L \geq b$ for all $b \in B$ (this just says that L is an upper bound for b) and such that if M is any other upper bound for B, then $L \leq M$ (this just says that L is $\leq$ any other upper bound).

Solution: The statement is simple. *If* $[a, b] \subseteq \mathbb{R}$ *is a closed interval, then* $[a, b]$ *is compact.* We prove it as follows. Let $\{U_i\}_{i \in I}$ be an open cover of $[a, b]$. We let B denote the set of elements $x \in [a, b]$ such that $[a, x]$ has a finite cover by open sets among the U_i. Note that a is in at least one U_j, and so there is an $x > a$ in the same U_j with $[a, x] \subseteq U_j$ (since each U_i is open). Thus the set B must contain that x since $[a, x]$ has a finite subcover, namely just one open set U_i. Hence B is not empty. By the fact, B has a least upper bound, call it y. By definition, for each $x < y$, we see that $[a, x]$ has a finite subcover by the U_i's, this uses the fact that y is the *least* upper bound of B.

Now, $y \in [a, b]$ so there exists some open set U_k in our cover so that $y \in U_k$. Then we see that $(y - \varepsilon, y + \varepsilon) \in U_k$ as well for some $\varepsilon > 0$ sufficiently small. Set $x = y - \varepsilon/2$, we can then choose a finite cover $\{U_1, \ldots, U_t\}$ of $[a, x]$ by hypothesis. But notice that $\{U_1, \ldots, U_t, U_k\}$ covers all the way to y, so $[a, y]$ has a finite cover. If $y = b$ we are done, but if $y < b$ the notice that $[a, y + \varepsilon/2]$ is also covered by $\{U_1, \ldots, U_t, U_k\}$. But that's impossible since it contradicts the fact that y was the upper bound for the set of $[a, x]$ such that $[a, x]$ has a finite cover by U_is.

17. Suppose that $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a continuous function. Prove that if $W \subseteq \mathbb{R}^n$ is a compact set, then $f(W) \subseteq \mathbb{R}^m$ is also compact.

Hint: Let $\{U_i\}$ be an open cover of $f(W)$ (presumably there are infinitely many U_i). You need to prove that there is a finite subcover of the U_i's.

Solution: This is quite easy, let $\{U_i\}_{i \in I}$ be a cover of $f(W)$. Then we claim that $\{f^{-1}(U_i)\}_{i \in I}$ form an open cover for W. Indeed, each $f^{-1}(U_i)$ is open by a previous assignment and so we just need to show they cover W. Choose $w \in W$, then $f(w) \in f(W)$ and so $f(w) \in U_j$ for some j. But then $w \in f^{-1}(U_j)$. This shows that each $w \in W$ is in at least one $\{f^{-1}(U_i)\}_{i \in I}$ which shows that those sets form an open cover of W. Since W is compact, we have a finite subcover $\{f^{-1}(U_1), \ldots, f^{-1}(U_k)\}$. We claim that $\{U_1, \ldots, U_k\}$ also form a finite subcover of $f(W)$. Indeed, choose $y \in f(W)$. Since everything in $f(W)$ is the image of something in W, we see that $y = f(x)$ for some $x \in W$. But then $x \in f^{-1}(U_j)$ for some $1 \leq j \leq k$ and so $y = f(x) \in U_j$. Finally now we see that $\{U_1, \ldots, U_k\}$ indeed cover W and the proof is complete.